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Entailment systems for stably locally compact locales

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Abstract

The category $\mathbf{SCFr}_U$ of stably continuous frames and preframe homomorphisms (preserving finite meets and directed joins) is dual to the Karoubi envelope of a category **Ent** whose objects are sets and whose morphisms $X \to Y$ are upper closed relations between the finite powersets $\mathscr{F}X$ and $\mathscr{F}Y$. Composition of these morphisms is the "cut composition" of Jung et al. that interfaces disjunction in the codomains with conjunctions in the domains, and thereby relates to their *multi-lingual sequent calculus.* Thus stably locally compact locales are represented by "entailment systems" $(X,\vdash)$ in which $\vdash$, a generalization of entailment relations, is idempotent for cut composition. Some constructions on stably locally compact locales are represented in terms of entailment systems: products, duality and powerlocales. Relational converse provides **Ent** with an involution, and this gives a simple treatment of the duality of stably locally compact locales. If A and B are stably continuous frames, then the internal preframe hom $A \pitchfork B$ is isomorphic to $\tilde{A} \otimes B$ where $\tilde{A}$ is the Hofmann–Lawson dual. For a stably locally compact locale X, the lower powerlocale of X is shown to be the dual of the upper powerlocale of the dual of X.

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1. Introduction

The goal of this paper is to develop a logical framework, analogous to the *information systems* used in presenting domains, but adapted for presenting arbitrary stably locally compact locales [6]. These are exactly the retracts of spectral (or coherent)

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locales, whose frames are the ideal completions of distributive lattices. The work develops the logical ideas of the *multi-lingual sequent calculus* of [10,13].

1.1. Information systems

Information systems were introduced [15] as a means for handling Scott (bounded complete algebraic) domains. Rather than working with the entire set of points, the information system uses a basis A of "token" points of which all the others are joins. The structure of the information system—sufficient to capture the structure of the domain up to isomorphism—then also comprises a unary "consistency" predicate Con on $\mathscr{F}A$, and a binary "entailment" relation $\vdash$ from $\mathscr{F}A$ to A (where $\mathscr{F}A$ is the finite powerset on A). $\mathrm{Con}(U)$ asserts that U is bounded above in the domain (and so has a join), while $U \vdash a$ asserts that U is consistent and its join is bigger than a in the domain. Along with the information systems also go "approximable mappings", relations from $\mathscr{F}A$ to $\mathscr{F}B$ (for B another information system) that describe Scott continuous maps between the corresponding domains.

The machinery there makes essential use of bounded completeness by using finite sets of tokens to denote their join. However, a similar notion has been used for algebraic domains in general, taking the tokens to be all compact (finite) points, and has even been extended (e.g. [16,21]) to continuous domains.

In fact we shall understand the phrase "information system" in a rather broad sense that includes localic approaches that use generators and relations for frames rather than the entire frames. An important example is [1], which describes spectral locales using generators and relations for the distributive lattices (of compact opens), and goes on to describe SFP domains in similar terms. There a token can be taken to be a generating (subbasic) compact open. (This can be compared with the subsequent [23], which takes the compact points of an SFP domain as the tokens.)

Reasons for using information systems in the broad sense can be seen in various mathematical contexts. They are used in the solution of recursive domain equations, taking least fixpoints in a cpo of countable information systems. (This also appears in topos guise in [23], with a topos whose points are the information systems.) In effective domain theory, the fact that domains are in general uncountable forces one to use bases instead. Similar to this is the situation in predicative type theory, where powersets are not admissible as legitimate sets and neither are most domains. Again, bases must be used.

Finally, we mention the *geometricity* constraints in the "topology-free space" approach of [23]. This exploits the fact that locales *do* have sufficient points if one admits generalized points—in the constructive set theories internal to arbitrary toposes. But one is then constrained to use reasoning that transfers well from one topos to another: explicitly, reasoning that is "geometric" in the sense of being preserved by inverse functors of geometric morphisms. That does not include ideal completion, power sets, or the construction of frames, and again one finds oneself having to use information systems.

Tacitly, we take "information system" to mean a geometric description of a locale. However, it is evident from the above that there are important similar notions in other mathematical contexts. In the present work it will bear a close relationship with logic and the sequent calculus.

1.2. Stably locally compact locales

Recall that a locale X is *stably locally compact* (often called *stably compact*) iff its frame ΩX is *stably continuous*—that is to say, it is continuous and if $a \ll b_i$ $(1 \leqslant i \leqslant n)$ then $a \ll \bigwedge_i b_i$. ($\ll$ is the way below relation.) Obviously it suffices to check the nullary ($n = 0$, saying that X is compact) and binary ($n = 2$) cases here. The stably locally compact locales are precisely the retracts of the spectral locales, i.e. those for which the frame is the ideal completion of a distributive lattice. For further details, see [6].

Amongst algebraic domains, the stably locally compact locales are the "$\frac{2}{3}$-SFP domains". These satisfy the condition that if U is a finite set of compact points then there is a finite set V of compact upper bounds of U that is complete as such: every upper bound of U is above some element of V. These are by no means all algebraic domains, but include the important classes of the SFP domains and the Scott domains. Analogous classes of continuous domains are also stably locally compact. On the other hand, stably locally compact locales also include all compact regular locales, and so encompass an important part of traditional topology.

One important view of stably locally compact locales is as ordered compact regular locales. This generalizes the Priestley duality by which a spectral locale is equivalent to an ordered Stone locale. The spatial version of this derives from Nachbin's theory of ordered compact Hausdorff spaces, and in a localic setting has been studied in [3,9,18] and [5] (which also describes some of the origins of the spatial theory). In particular, [18] showed how to deal with stably locally compact locales by the use of "Hausdorff systems", splitting idempotent preframe endomorphisms of compact regular frames. The present work uses a similar idea with free frames.

Once a set X of generators is given for a distributive lattice L, we have L as a homomorphic image of the free distributive lattice $\mathbf{DL}\langle X \rangle$. It follows that the ideal completion $\mathrm{Idl}(L)$, the frame for the corresponding spectral locale (the spectrum of L), is a homomorphic image of the free frame $\mathbf{Fr}\langle X \rangle$. Since any stably locally compact locale Y is a retract of a spectral locale, it follows that its frame ΩY is a homomorphic image of $\mathbf{Fr}\langle X \rangle$ and hence can be described by a nucleus on $\mathbf{Fr}\langle X \rangle$. We should like to take X as the token set.

But any frame is a homomorphic image of a free frame! Simply looking at nuclei on free frames does not get us down to stable local compactness. We shall modify the nuclei in two ways.

First, we shall impose Scott continuity. (Scott continuous nuclei have been described in [5].) This is key to describing the nuclei in a finitary way in terms of the tokens, but unfortunately it is too restrictive. A Scott continuous nucleus is one for which the sublocale embedding i is *perfect*, in other words the right adjoint of the inverse image homomorphism i^* is Scott continuous. But a perfect sublocale of a spectral locale is again spectral, essentially because i^* preserves compactness. (For any perfect map f, f^* preserves way below.)

Second, to compensate we shall generalize slightly the notion of nucleus by allowing them not to be inflationary. We shall define a *quasinucleus* on a frame A to be a function $j: A \to A$ that preserves finite meets and directed joins (i.e. it is a *preframe homomorphism*), and is idempotent. We find then that the set A_j of fixpoints of j is also a frame, just as for a nucleus. The main loss in this generalization is that the function $A \to A_j$, though it is a preframe homomorphism, need not preserve finite joins, so A_j is not in general a frame homomorphic image of A.

If we work in the category of stably continuous frames and preframe homomorphisms then we find it is closed under this splitting of idempotents, and moreover that every stably continuous frame can be obtained in this way by splitting an idempotent on a free frame. Our information system techniques rely on a concrete description (cf. approximable mappings) of preframe homomorphisms between free frames.

1.3. Logic and entailment

Spectral locales (and perfect maps) are dual to distributive lattices, specifically their lattices of compact opens, and this provides algebraic techniques for handling spectral locales via the lattices. In particular we can present them by generators and relations.

Suppose X is a given set of generators. We can reduce a system of relations (in the sense of equations between distributive lattice words in X) to a simple form as follows. First, an equation $e_1 = e_2$ can be replaced by a pair of inequations $e_1 \leqslant e_2$ and its reverse. Next, we can reduce e_1 and e_2 to, respectively, disjunctive and conjunctive form, and then the inequation $e_1 \leqslant e_2$ can be replaced by a finite set of inequations of the form $\bigwedge U \leqslant \bigvee V$ where U and V are finite subsets of X. Writing $U \vdash V$ for $\bigwedge U \leqslant \bigvee V$, it follows that any presentation by generators and relations can be transformed to an equivalent one described by a relation $\vdash \subseteq \mathscr{F}X \times \mathscr{F}X$, where $\mathscr{F}X$ is the finite powerset of X. Every such $\vdash$ presents a distributive lattice

$$\mathrm{DL}\,\langle X \mid \bigwedge U \leqslant \bigvee V \ (U \vdash V)\rangle$$

To put this another way, congruences on the free distributive lattice $\mathrm{DL}\langle X\rangle$ can be described using relations $\vdash$. This has been sharpened [4]: congruences correspond bijectively to relations $\vdash$ that satisfy the following three rules:

$$\frac{}{s \vdash s} \qquad \text{(reflexivity)}$$

$$\frac{U \vdash V}{U, U' \vdash V, V'} \qquad \text{(weakening)}$$

$$\frac{U \vdash s, V \qquad U, s \vdash V}{U \vdash V} \qquad \text{(cut)}$$

(Comma here denotes union, and the letter s denotes a singleton subset.)

Since every stably locally compact locale is a retract of a spectral locale, one can describe each one as a distributive lattice with extra structure. Examples are *proximity lattices* [17] and *strong proximity lattices* [12], distributive lattices with an extra "strong" ordering $<$ satisfying various conditions. In fact these are already information systems in our broad (geometric) sense, and strong proximity lattices have been exploited in [26] to give a geometric account of sheaves over stably locally compact locales. The distributive lattice itself can be presented by an entailment system. In particular, a *coherent sequent calculus* ([10,13]; see also [11]) has a structure $(L; \wedge, \vee, \top, \bot; \Vdash)$ in which

(1) $\Vdash$ is a binary relation on $\mathscr{F}L$, with weakening as in an entailment system.
(2) Reflexivity is dropped.
(3) Cut is replaced by a rule that $\Vdash$ is idempotent under a *cut composition* $\dagger$ that we shall discuss in more detail later. (The cited papers denote it by $\circ$, but we use a different symbol to avoid confusion with relational composition.) Symbolically,

$$\Vdash \dagger \Vdash = \Vdash$$

This idempotence is equivalent to cut in the presence of reflexivity, but not otherwise. (See Section 6.1.)
(4) $\wedge$, $\vee$, $\top$ and $\bot$ are two binary operations and two constants on L.
(5) There are rules to ensure that in an entailment $S \Vdash T$, S is equivalent to the conjunction ($\wedge$) of its elements and T to the disjunction ($\vee$).
(6) $\Vdash$ "has interpolants" in the sense that
- if $\{\phi\} \cup S \Vdash T$ then there is some ϕ' with $\{\phi\} \Vdash \{\phi'\}$ and $\{\phi'\} \cup S \Vdash T$; and
- if $S \Vdash \{\phi\} \cup T$ then there is some ϕ' with $\{\phi'\} \Vdash \{\phi\}$ and $S \Vdash \{\phi'\} \cup T$.

One senses that in a strong proximity lattice the information given in $<$ overlaps to some extent with that in the lattice. Analogously, in a coherent sequent calculus the lattice operations appear twice: $\wedge$ appears both explicitly and implicitly (as set union to the left of $\Vdash$), and $\vee$ is similar. We shall show how a single "strong" entailment $\vdash$ with weakening and cut idempotence can contain all the information needed to present a stably locally compact locale. This develops results in [13] by which the behaviour of $\wedge$ and $\vee$ is generated inductively, but at the same time we dispense with the interpolants of condition (6) above.

To get some idea of the problem that cut composition is addressing, think of $U \vdash W$ as meaning $\bigwedge U \ll \bigvee W$ and consider what is going to correspond to transitivity of $\ll$. We certainly do not have transitivity of $\vdash$ itself; if $U \vdash V$ and $V \vdash W$ then this means $\bigwedge U \ll \bigvee V$ and $\bigwedge V \ll \bigvee W$. In one place "$V$" stands for $\bigwedge V$, while in the other it is $\bigvee V$. We can only deduce $U \vdash W$ if V is a singleton. Let us think more generally, and suppose $U \vdash V_i$ for $i \in I$, meaning $\bigwedge U \ll \bigwedge_{i \in I} \bigvee V_i$. By distributivity, we can transform this meet of joins into a join of meets: it is (modulo some constructive care) $\bigvee \{\bigwedge \mathrm{Im}\,\gamma \mid \gamma$ a choice function of $\{V_i \mid i \in I\}\}$. (You can probably appreciate roughly how this works if you imagine actually working out such a distributivity.) This is then way below $\bigvee W$ if for every γ we have $\mathrm{Im}\,\gamma \vdash W$. To summarize, suppose

- for every $i \in I$ (I finite) we have $U \vdash V_i$,
- for every $j \in J$ (J finite) we have $V_j \vdash W$, and
- for every choice function $\gamma \in \prod_i V_i$ there is some $j \in J$ such that $\mathrm{Im}\,\gamma \supseteq V_j$.

Then $U \vdash W$. Jung et al. [10] express this principle by defining a "cut composition" operator "$\circ$" (which we shall denote "$\dagger$") on the relations such that if the above situation we replace $V_j \vdash W$ by $V_j \vdash' W$, then we deduce $U(\vdash \dagger \vdash')W$. Then an entailment relation $\vdash$ is expected to satisfy $\vdash = \vdash \dagger \vdash$. Clearly it is closely bound up with distributivity and distributive lattice duality.

Note, however, that the cut rule is not purely a special case of this general principle. To deduce cut we need also to assume something like reflexivity, as follows. Suppose we have $U \vdash s, V$ and $U, s \vdash V$. By reflexivity we also have $U \vdash \{u\}$ for each $u \in U$. We can now apply the general principle, with the V_is being $\{s\} \cup V$ and all the singletons $\{u\}$. A choice γ must choose all those u's, so its image is either $U \cup \{s\}$ or $U \cup \{v\}$ for some $v \in V$. The sequent $U, s \vdash V$ is given, and $U, v \vdash V$ follows by reflexivity (and weakening). Hence we can deduce that $U \vdash V$.

Thus, as we shall see, in replacing the cut rule by the generalized principle we also decouple ourselves from the reflexivity rule. This is essential in moving from spectral locales to stably locally compact (from an algebraic frame to a continuous one), because we shall want to use the way below relation $\ll$ and that is not reflexive. (However, [13] and [10] manage to retain the cut rule by their assumption of interpolants (condition (6) above).)

Our arguments will be constructive throughout. Indeed, we take care to ensure that the definitions relating to entailment systems should be geometric (see [23]). "Finite" will always mean *Kuratowski finite*, and we write $\mathscr{F}X$ for the set of finite subsets of X. (See [8], where our $\mathscr{F}X$ is denoted $K(X)$.)

1.4. Summary

Our exposition is in four stages. In the first stage (Section 2), we describe properties of the symmetric monoidal category $\mathbf{SCFr}_U$ of stably continuous frames and preframe homomorphisms. In the second stage (Section 3), we discuss how these properties interact with splitting idempotents. After some preparation with auxiliary results (Section 4) on choices and free distributive lattices, the third stage (Section 5) shows how the properties appear in the full subcategory of free frames. In the fourth stage (Section 6), we put these together to use "entailment systems" to describe arbitrary stably locally compact locales. In Section 7 we show as an application how to exploit the symmetry of the calculus to describe self-duality and monoidal closure in $\mathbf{SCFr}_U$ (and hence of stably locally compact locales). Section 8 then describes the powerlocales in terms of entailment systems.

2. Products of stably locally compact locales

We shall denote by **SLCLoc** the category of stably locally compact locales and continuous maps (dual to stably continuous frames and frame homomorphisms); however, for various reasons we shall find it convenient to work within a larger category $\mathbf{SLCLoc}_U$ dual to the category $\mathbf{SCFr}_U$ of stably continuous frames and *preframe* homomorphisms (preserving finite meets and directed joins). This larger category is in many ways simpler to work with than **SLCLoc**, it has some good categorical features—for instance it is monoidal closed and self-dual—and it includes morphisms corresponding to some of our technical features such as the quasinuclei. Having developed those properties it is then easier for us to identify (in Theorem 42) **SLCLoc** as a subcategory of $\mathbf{SLCLoc}_U$. For comparison, if one looks at [21], it is actually easier there to study first the "lower approximable semimappings" between information systems (corresponding to maps to the *lower* powerlocale), and then to specialize to the continuous maps. The spatial version of the category $\mathbf{SLCLoc}_U$ has been studied (classically) in [11].

Preframe homomorphisms $\Omega Y \to \Omega X$ correspond to maps $X \to P_U Y$, where P_U is the upper powerlocale monad. (This is immediate from the definition of P_U: the frame $\Omega P_U X$ is the free frame over ΩX qua preframe.) We shall call a map $X \to P_U Y$ an *upper relation* from X to Y. P_U is the localic version of the Smyth powerdomain, and so the upper relations correspond to what are often identified in domain theory with the "demonic" non-deterministic maps between domains. Also **SLCLoc** is closed under P_U—this is easily seen for spectral locales, and then one argues by retracts. It follows that $\mathbf{SLCLoc}_U$ is the Kleisli category for P_U restricted to **SLCLoc**.

For comparison with the terminology of [13], our $\mathbf{SCFr}_U$ is his ASL. (He calls a stably continuous frame an *arithmetic lattice*, or, the same thing but with preframe homomorphisms understood as morphisms, an *arithmetic semilattice*.) Our $\mathbf{SLCLoc}_U$ is his $\mathsf{StCp}_{\mathscr{K}}$.

$\mathbf{SCFr}_U$ has a monoidal product. The category **PreFr** of preframes has a tensor product $\otimes$, which, when restricted to frames, coincides with the frame coproduct [9]. We write $(a,b) \mapsto a \odot b$ for the universal preframe bimorphism $\Omega X \times \Omega Y \to \Omega(X \times Y)$; it is different from the suplattice bimorphism $(a,b) \mapsto a \times b$, with $a \odot b = a \times 1 \vee 1 \times b$ (so a point (x,y) is in $a \odot b$ iff either x is in a or y is in b). Conversely, we can define $a \times b = a \odot 0 \wedge 0 \odot b$. The class of spectral locales is closed under products, and it follows that so too is the class of stably locally compact locales. Hence $\mathbf{SCFr}_U$ is closed under $\otimes$.

The unit of the monoidal product is the subobject classifier Ω (which is also the free preframe on one generator, **false**). The isomorphism $\Omega \otimes A \to A$ is given by

$$p \odot a \mapsto \bigvee\nolimits^{\uparrow} \{\{a\} \cup \{1 | p\}\}.$$

It is an instructive exercise to show that this is a bimorphism, and that it gives an inverse to the homomorphism $a \mapsto \mathbf{false} \odot a$.

For future reference, we shall need the following lemma.

Lemma 1. *Let A and B be two stably continuous frames. Suppose $a \ll a'$ in A, and $b \ll b'$ in B. Then if $a' \times b' \leqslant c \odot d$ we have either $a \ll c$ or $b \ll d$.*

Proof. Define $\theta : A \times B \to \Omega$ by

$$\theta(x,y) \quad \text{iff} \quad a \ll x \text{ or } b \ll y.$$

This is a preframe bimorphism, and so defines a preframe homomorphism $f : A \otimes B \to \Omega$, $f(x \odot y)$ iff $\theta(x,y)$. Since $a' \times b' = a' \odot 0 \wedge 0 \odot b'$, we see that $f(a' \times b')$ holds, and hence so does $f(c \odot d)$. $\square$

3. Splitting idempotents in $\mathbf{SCFr}_U$

In this section we investigate the splitting of idempotents in $\mathbf{SCFr}_U$ ("quasinuclei") and show that every stably continuous frame can be got by splitting a quasinucleus on a free frame. More precisely, $\mathbf{SCFr}_U$ is equivalent to the Karoubi envelope of its full subcategory on free frames.

Definition 2. Let A be a stably continuous frame. A *quasinucleus* on A is an idempotent preframe endomorphism on A.

There are two differences between a quasinucleus and an ordinary nucleus: the quasinucleus is required to preserve directed joins, but it is not required to be inflationary. However, we shall show that essentially the same construction as for nuclei enables us to construct new frames.

Given a quasinucleus j on A, we write A_j for the preframe of fixpoints of j. This splits j, with preframe homomorphisms $j_{\text{in}}: A_j \to A$ and $j_{\text{out}}: A \to A_j$. If j is inflationary (hence a nucleus), then j_{out} is a frame homomorphism, but this is not so in general.

Theorem 3. *Let A be a stably continuous frame and j a quasinucleus on it. Then the poset A_j of fixpoints of j is a stably continuous frame.*

Proof. First, as equalizer of j and Id_A, A_j is a subpreframe of A.

Next, we show that A_j has joins, and meet distributes over them. If $S \subseteq A_j$, its join is

$$\bigvee S = j_{\text{out}}\left(\bigvee\{j_{\text{in}}(a) \mid a \in S\}\right).$$

For distributivity,

$$\begin{aligned}\bigvee\{b \wedge a \mid a \in S\} &= j_{\text{out}}\left(\bigvee\{j_{\text{in}}(b \wedge a) \mid a \in S\}\right)\\ &= j_{\text{out}}\left(j_{\text{in}}(b) \wedge \bigvee\{j_{\text{in}}(a) \mid a \in S\}\right)\\ &= b \wedge \bigvee S.\end{aligned}$$

Hence A_j is a frame.

Next we show that $a \ll b$ in A_j iff $a \leqslant j_{\text{out}}(b')$ for some $b' \ll j_{\text{in}}(b)$. For the $\Leftarrow$ direction, suppose $b \leqslant \bigvee^{\uparrow} S$ where S is a directed subset of A_j. Then $j_{\text{in}}(b) \leqslant \bigvee^{\uparrow}\{j_{\text{in}}(c) \mid c \in S\}$, so $b' \leqslant$ some $j_{\text{in}}(c)$ and $a \leqslant c$. For the $\Rightarrow$ direction, we have $j_{\text{in}}(b) = \bigvee^{\uparrow}\{b' \mid b' \ll j_{\text{in}}(b)\}$, and so $b = \bigvee^{\uparrow}\{j_{\text{out}}(b') \mid b' \ll j_{\text{in}}(b)\}$.

Note also in the above that for each b' we have $j_{\text{out}}(b') \ll b$, and it follows that b is a directed join of elements way below it. Hence A_j is continuous.

For stability, suppose $a, b_i \in A_j$ $(1 \leqslant i \leqslant n)$, $a \leqslant j_{\text{out}}(b_i')$ and $b_i' \ll j_{\text{in}}(b_i)$. Then $a \leqslant j_{\text{out}}(\bigwedge_{i=1}^n b_i')$ and $\bigwedge_{i=1}^n b_i' \ll j_{\text{in}}(\bigwedge_{i=1}^n b_i)$, so $a \ll \bigwedge_{i=1}^n b_i$. $\square$

Lemma 4. *Let j be a quasinucleus on A. If $a \ll j_{\text{in}}(b)$ in A, then $j_{\text{out}}(a) \ll b$ in A_j.*

Proof. This follows from the characterization of $\ll$ in A_j that was given in the proof of Theorem 3. □

The preframe homomorphism $j_{\text{out}} : A \to A_j$ is not in general a frame homomorphism: for $a, b \in A$ we do not necessarily have $j(a \vee b) = j(j(a) \vee j(b))$. Thus our construction does not necessarily construct sublocales.

Example 5. Let $A = \{0, a \wedge b, a, b, a \vee b, 1\}$ be the free distributive lattice on two generators a and b. For simplicity, we assume classical logic here so that A is already its own ideal completion, the free frame on $\{a, b\}$. Define a quasinucleus j by

$$\begin{aligned} 1, a \vee b &\mapsto 1 \\ a, b, a \wedge b &\mapsto a \\ 0 &\mapsto 0. \end{aligned}$$

j is neither inflationary nor deflationary. Its fixpoints are $A_j = \{0, a, 1\}$. The function $j_{\text{out}} : A \to A_j$ takes both a and b to a, but takes $a \vee b$ to 1 and so does not preserve finite joins.

Proposition 6. *Let A be a stably continuous frame. Then there is a quasinucleus j on the free frame $\mathbf{Fr}\langle A \rangle$ such that A is isomorphic to $\mathbf{Fr}\langle A \rangle_j$.*

Proof. Let the frame homomorphism $\alpha : \mathbf{Fr}\langle A \rangle \to A$ be the structure map for A as frame. Define $\beta : A \to \mathbf{Fr}\langle A \rangle$ by

$$\beta(a) = \bigvee\nolimits^{\uparrow} \{\phi \in \mathbf{DL}\langle A \rangle \mid \alpha(\phi) \ll a\}.$$

Here $\mathbf{DL}\langle A \rangle$ denotes the free distributive lattice on A; $\mathbf{Fr}\langle A \rangle$ is its ideal completion. β is a preframe homomorphism with $\alpha \circ \beta = \mathrm{Id}_A$, and our quasinucleus j is $\beta \circ \alpha$. □

Thus every stably continuous frame can be got by splitting a quasinucleus on a free frame. We can express this more precisely using a Karoubi envelope. Let $\mathbf{FreeFr}_U$ be the full subcategory of $\mathbf{SCFr}_U$ on the free frames.

Theorem 7. $\mathbf{SCFr}_U$ *is equivalent to the Karoubi envelope* $\mathbf{Kar}(\mathbf{FreeFr}_U)$.

Proof. The objects of the Karoubi envelope are pairs (A, j), where A is an object in $\mathbf{FreeFr}_U$ and j is an idempotent endomorphism on A. A morphism $(A, j) \to (B, k)$ is a morphism $f : A \to B$ in $\mathbf{FreeFr}_U$ such that $f = j; f; k$.

Since every idempotent in $\mathbf{SCFr}_U$ splits, the full and faithful embedding of $\mathbf{FreeFr}_U$ in $\mathbf{SCFr}_U$ factors via a full and faithful embedding of $\mathbf{Kar}(\mathbf{FreeFr}_U)$ in $\mathbf{SCFr}_U$, and we have seen that it is essentially surjective on objects. □

Proposition 8. *Let A and B be stably continuous frames, with quasinuclei j and k respectively. Then*

$$(A \otimes B)_{j\otimes k} \cong A_j \otimes B_k.$$

Proof. $j_{\mathrm{out}} \otimes k_{\mathrm{out}}$ and $j_{\mathrm{in}} \otimes k_{\mathrm{in}}$ split $j \otimes k$. $\square$

4. Some auxiliary results

We gather in this section some auxiliary results relating to distributivity and free distributive lattices. We take care to give a constructive treatment that is *geometric*, i.e. preserved by inverse image functors of geometric morphisms between toposes. Our treatment is also intended to be predicative.

Note that "finite" will always mean Kuratowski finite, so that our finite powerset $\mathscr{F}X$ (often denoted $K(X)$) is the free semilattice on X. Universal quantification bounded over a finite set is geometric, unlike more general universal quantifications. We note here some constructive results that we shall use.

Proposition 9 (Simple $\mathscr{F}$-induction, Vickers [23]). *Let $\phi(S)$ be a predicate on $\mathscr{F}X$ such that $\phi(\emptyset)$ (base case), and if $\phi(S)$ then $\phi(\{x\}\cup S)$ for all $x:X$ (induction step). Then $\phi(S)$ holds for all S.*

Proposition 10 ($\mathscr{F}$-recursion, Vickers [23]). *Let $f: X \times Y \to Y$ satisfy*
(1) $\forall x, x', y.\ f(x, f(x', y)) = f(x', f(x, y))$
(2) $\forall x, y.\ f(x, f(x, y)) = f(x, y)$
Then there is a unique $g: \mathscr{F}X \times Y \to Y$ such that

$$\forall y.\ g(\emptyset, y) = y$$
$$\forall x, y.\ g(\{x\}, y) = f(x, y)$$
$$\forall S, T, y.\ g(S \cup T, y) = g(S, g(T, y))$$

Proposition 11 (Johnstone [7]). *Let ϕ be a predicate on X, let $S \in \mathscr{F}X$, and suppose $\forall x \in S.\ (\phi(x) \vee \psi(x))$. Then either $\forall x \in S.\ \phi(x)$ or $\exists x \in S.\ \psi(x)$*

4.1. Choices

For general application of distributive laws and the definition of cut composition we need choice functions, but for constructivist reasons we replace these by *total choice relations*. To see why, let X be a set and $\mathscr{V}$ a finite set of finite subsets of X. A choice function for $\mathscr{V}$ is a function $\gamma: \mathscr{V} \to \bigcup \mathscr{V}$ such that $\gamma(V) \in V$ for every $V \in \mathscr{V}$. Since $\mathscr{V}$ is finite, the graph of γ must be a finite subset of $\mathscr{V} \times \bigcup \mathscr{V}$, but unfortunately the set of such γ's is not a geometrically definable subset of $\mathscr{F}(\mathscr{V} \times \bigcup \mathscr{V})$.

This is because the single-valuedness property of a function,

$$\forall V, v, v'.\ (V\gamma v \wedge V\gamma v' \rightarrow v = v'),$$

is not geometric. To make it so we should need decidable equality on X, when the property can be expressed as

$$\forall (V, v) \in \gamma.\ \forall (V', v') \in \gamma.\ (V \neq V' \vee v = v').$$

However, it turns out that single-valuedness is not needed for our applications. We just need *at least one* $\gamma(V) \in V$ for every V, so that γ needs to be a total relation but not necessarily single valued.

Definition 12. Let X be a set and $\mathscr{V} \in \mathscr{F}\mathscr{F}X$. A *choice* for $\mathscr{V}$ is some $\gamma \in \mathscr{F}(\mathscr{V} \times \bigcup \mathscr{V})$ satisfying

- $\forall V \in \mathscr{V}.\ \exists v \in V.\ (V, v) \in \gamma$
- $\forall (V, v) \in \gamma.\ v \in V$

The universal quantifications here are finitely bounded, as required for geometricity.

Definition 13. We write $\mathrm{Ch}(\mathscr{V})$ for the set of choices of $\mathscr{V}$, and $\mathrm{Im}\,\gamma$ for the *image* of γ, i.e. its direct image under the second projection to $\bigcup \mathscr{V}$ (note that $\mathrm{Im}\,\gamma$ is finite).

Proposition 14. *Let X be a set and $\mathscr{V} \in \mathscr{F}\mathscr{F}X$. Then $\mathrm{Ch}(\mathscr{V})$ is finite.*

Proof. This has been proved in [24], so we just sketch the proof here. We use $\mathscr{F}$-recursion (Proposition 10) to implement a function $\mathrm{Ch} : \mathscr{F}\mathscr{F}X \rightarrow \mathscr{F}\mathscr{F}(\mathscr{F}X \times X)$ whose specification is that $\gamma \in \mathscr{F}(\mathscr{F}X \times X)$ is in $\mathrm{Ch}(\mathscr{V})$ iff it is a choice for $\mathscr{V}$ as defined above.

We define

$$\begin{aligned} \mathrm{Ch}(\emptyset) &= \{\emptyset\}, \\ \mathrm{Ch}(\mathscr{V} \cup \{U\}) &= \{\gamma \cup (\{U\} \times S) \mid \gamma \in \mathrm{Ch}(\mathscr{V}) \text{ and } S \in \mathscr{F}^{+}U\}, \end{aligned}$$

where $\mathscr{F}^{+}U$ denotes the set of nonempty finite subsets of U. (Note that emptiness is a decidable property of finite sets.) The proof obligations of $\mathscr{F}$-recursion are easy. An $\mathscr{F}$-induction proof then proves correctness, i.e. that the function does implement its specification. □

Proposition 15. *Suppose ϕ is a predicate on X, $\mathscr{U} \in \mathscr{F}\mathscr{F}X$ and*

$$\forall U \in \mathscr{U}.\ \exists u \in U.\ \phi(u).$$

Then there is some choice γ for $\mathscr{U}$ whose image elements all satisfy ϕ.

Proof. Use induction on $\mathscr{U}$. □

The following lemma is quite fundamental.

Lemma 16 (Diagonalization Lemma). *Let X be any set and ϕ a predicate on it. Let $\mathscr{V} \in \mathscr{F}\mathscr{F}X$ be such that $\forall \gamma \in \mathrm{Ch}(\mathscr{V})$. $\exists v \in \mathrm{Im}\,\gamma$. $\phi(v)$. Then there is some V in $\mathscr{V}$ such that $\forall v \in V$. $\phi(v)$.*

Proof. *Classically* one would use a diagonalization argument as follows. If there is no such V, then in every V we can choose an element v such that $\neg\phi(v)$, and this gives a choice γ for $\mathscr{V}$ whose image does not meet ϕ—contradiction. However, we must avoid the proof by contradiction, and we use simple $\mathscr{F}$-induction on $\mathscr{V}$.

If $\mathscr{V} = \emptyset$ then $\mathrm{Ch}(\mathscr{V}) = \{\emptyset\}$ and the hypothesis implies that ϕ meets $\emptyset$—contradiction.

Now suppose $\mathscr{V} = \{V\} \cup \mathscr{V}'$ with the result holding for $\mathscr{V}'$. If $v \in V$ then every choice γ' for $\mathscr{V}'$ gives us a choice $\{(V, v)\} \cup \gamma'$ for $\mathscr{V}$, and so either the image of γ' meets ϕ or $\phi(v)$ holds. By Proposition 11 it follows that either $\forall \gamma' \in \mathrm{Ch}(\mathscr{V}')$. $\exists v' \in \mathrm{Im}\,\gamma'$. $\phi(v')$ or $\exists \gamma' \in \mathrm{Ch}(\mathscr{V}')$. $\phi(v)$. In the first case by induction we have some V' in $\mathscr{V}'$ such that $\forall v' \in V'$. $\phi(v')$, while in the second case we have $\phi(v)$. Hence we have

$$\forall v \in V.\ (\phi(v) \vee \exists V' \in \mathscr{V}'.\ \forall v' \in V'.\ \phi(v')).$$

It follows that either $\forall v \in V$. $\phi(v)$ or $\exists v \in V$. $\exists V' \in \mathscr{V}'$. $\forall v' \in V'$. $\phi(v')$. Either way, we get V' in $\{V\} \cup \mathscr{V}'$ such that $\forall v' \in V'$. $\phi(v')$ as required. $\square$

Corollary 17. *Let $\mathscr{V} \in \mathscr{F}\mathscr{F}X$ and let $\mathscr{V}' = \{\mathrm{Im}\,\gamma \mid \gamma \in \mathrm{Ch}(\mathscr{V})\}$. Then $\forall \delta \in \mathrm{Ch}(\mathscr{V}')$. $\exists V \in \mathscr{V}$. $V \subseteq \mathrm{Im}\,\delta$.*

Proof. Given δ, define a predicate ϕ on X by $\phi(x)$ iff $x \in \mathrm{Im}\,\delta$. If $\gamma \in \mathrm{Ch}(\mathscr{V})$, then δ chooses an element of $\mathrm{Im}\,\gamma$ that is also in $\mathrm{Im}\,\delta$. We can thus apply the Diagonalization Lemma. $\square$

For us, the most important use of choices lies in expressing generalized distributivity in distributive lattices.

Theorem 18. *Let L be a distributive lattice and let $\mathscr{V} \in \mathscr{F}\mathscr{F}L$. Then*

$$\bigvee_{V \in \mathscr{V}} \bigwedge V = \bigwedge_{\gamma \in \mathrm{Ch}(\mathscr{V})} \bigvee \mathrm{Im}\,\gamma.$$

Proof. See [24]. The proof is by induction on $\mathscr{V}$. $\square$

Of course, there is a dual result in which meets and joins are interchanged.

4.2. *The free distributive lattice* DL⟨X⟩

We now investigate the concrete structure of the free distributive lattice DL⟨X⟩ on generators X.

Clearly

$$\mathrm{DL}\langle X\rangle \cong \mathrm{DL}\langle \wedge\text{-semilattice}\langle X\rangle \text{ (qua } \wedge\text{-semilattice)}\rangle.$$

We put together some known results.

Proposition 19. (1) *Let A be a meet semilattice. Then*

$$\mathrm{DL}\langle A \text{ (}\mathit{qua}\ \wedge\text{-}\mathit{semilattice}\text{)}\rangle \cong \vee\text{-}\mathit{semilattice}\langle A \text{ (}\mathit{qua\ poset}\text{)}\rangle.$$

(2) *Let P be a poset. Then*

$$\vee\text{-}\mathit{semilattice}\langle P \text{ (}\mathit{qua\ poset}\text{)}\rangle \cong \mathscr{F}P/\sqsubseteq_L$$

where $U \sqsubseteq_L V$ iff $\forall u \in U.\ \exists v \in V.\ u \leqslant v$.

(3) *Let P be a poset. Then*

$$\wedge\text{-}\mathit{semilattice}\langle P \text{ (}\mathit{qua\ poset}\text{)}\rangle \cong \mathscr{F}P/\sqsubseteq_U$$

where $U \sqsubseteq_U V$ iff $\forall v \in V.\ \exists u \in U.\ u \leqslant v$.

(4) *Let X be a set. Then*

$$\mathrm{DL}\langle X\rangle \cong \mathscr{F}\mathscr{F}X/\leqslant$$

where $\mathscr{U} \leqslant \mathscr{V}$ iff $\forall U \in \mathscr{U}.\ \exists V \in \mathscr{V}.\ U \supseteq V$.

Proof. 1. This follows the same lines as (replacing distributive lattices by frames and $\vee$-semilattices by suplattices) the proof of the coverage theorem in [2]. Taking $L = \vee\text{-semilattice}\langle A \text{ (qua poset)}\rangle$, one can define a $\vee$-bimorphism on L that extends $\wedge$ on A, and then show that it is meet on L. This shows that L is a distributive lattice. It is then not hard to show that L has the right universal property.

2. One first shows that $\cup$ in $\mathscr{F}P$ is a join with respect to the preorder $\sqsubseteq_L$. Then the universal property is not hard.

3. This is dual to (2).

4. By (3) (applied to X as a discrete poset), the free meet semilattice on X is $\mathscr{F}X/\supseteq$. Now apply (1) and (2). □

In this we see every element of $\mathrm{DL}\langle X\rangle$ expressed in disjunctive form, as a join of meets of generators. But a dual result also holds in which every element is expressed in conjunctive form, as a meet of joins of generators. We introduce some notation to describe this.

Definition 20. Let X be a set.

(1) We define a function $\mathscr{U} \mapsto \phi_{\mathscr{U}}$ from $\mathscr{F}\mathscr{F}X$ to $\mathrm{DL}\langle X\rangle$ by

$$\phi_{\mathscr{U}} = \bigvee_{U \in \mathscr{U}} \bigwedge U$$

(2) We write $x \mapsto \tilde{x}$ for the lattice homomorphism from $\mathrm{DL}\langle X\rangle$ to $\mathrm{DL}\langle X\rangle^{\mathrm{op}}$ that is the identity on X.

The conjunctive form is

$$\phi_{\mathscr{U}} = \bigvee_{U \in \mathscr{U}} \bigwedge U = \bigwedge_{\gamma \in \mathrm{Ch}(\mathscr{U})} \bigvee \mathrm{Im}\,\gamma = \widetilde{\phi_{\mathscr{V}}}$$

where $\mathscr{V} = \{\mathrm{Im}\,\gamma \mid \gamma \in \mathrm{Ch}(\mathscr{U})\}$.

We see that

$$\phi_{\mathscr{U}} \leqslant \phi_{\mathscr{V}} \quad \text{iff} \quad \forall U \in \mathscr{U}.\ \exists V \in \mathscr{V}.\ U \supseteq V.$$

However, it turns out useful to investigate the relations $\phi_{\mathscr{U}} \leqslant \widetilde{\phi_{\mathscr{V}}}$ and $\widetilde{\phi_{\mathscr{U}}} \leqslant \phi_{\mathscr{V}}$. This is because they are symmetric—$\phi_{\mathscr{U}} \leqslant \widetilde{\phi_{\mathscr{V}}}$ iff $\phi_{\mathscr{V}} \leqslant \widetilde{\phi_{\mathscr{U}}}$ and similarly for the other one.

Proposition 21. *Let X be a set, and let $\mathscr{U}, \mathscr{V} \in \mathscr{F}\mathscr{F}X$. Then the following are equivalent.*

(1) $\phi_{\mathscr{U}} \leqslant \widetilde{\phi_{\mathscr{V}}}$.

(2) $\phi_{\mathscr{V}} \leqslant \widetilde{\phi_{\mathscr{U}}}$.

(3) $\forall U \in \mathscr{U}.\ \forall V \in \mathscr{V}.\ U$ *meets* V.

Proof. $(1) \Leftrightarrow (2)$ is obvious because $\tilde{}$ is an involution.

$(1) \Leftrightarrow (3)$: Since $\phi_{\mathscr{U}} = \bigvee_{U \in \mathscr{U}} \bigwedge U$ and $\widetilde{\phi_{\mathscr{V}}} = \bigwedge_{V \in \mathscr{V}} \bigvee V$, we see that $\phi_{\mathscr{U}} \leqslant \widetilde{\phi_{\mathscr{V}}}$ iff for every $U \in \mathscr{U}$ and $V \in \mathscr{V}$ we have $\bigwedge U \leqslant \bigvee V = \bigvee_{v \in V} \{v\}$. This holds iff $U \supseteq \{v\}$ for some $v \in V$, i.e. U meets V. $\square$

Proposition 22. *Let X be a set, and let $\mathscr{U}, \mathscr{V} \in \mathscr{F}\mathscr{F}X$. Then the following are equivalent.*

(1) $\widetilde{\phi_{\mathscr{U}}} \leqslant \phi_{\mathscr{V}}$.

(2) $\widetilde{\phi_{\mathscr{V}}} \leqslant \phi_{\mathscr{U}}$.

(3) $\forall \gamma \in \mathrm{Ch}(\mathscr{U}).\ \exists V \in \mathscr{V}.\ V \subseteq \mathrm{Im}\,\gamma$.

(4) $\forall \delta \in \mathrm{Ch}(\mathscr{V}).\ \exists U \in \mathscr{U}.\ U \subseteq \mathrm{Im}\,\delta$.

(5) $\forall \gamma \in \mathrm{Ch}(\mathscr{U}).\ \forall \delta \in \mathrm{Ch}(\mathscr{V}).\ \mathrm{Im}\,\gamma$ *meets* $\mathrm{Im}\,\delta$.

Proof. $(1) \Leftrightarrow (2)$ is obvious.

$(3) \Leftrightarrow (5)$ follows from the Diagonalization Lemma, and $(4) \Leftrightarrow (5)$ by symmetry.

$(1) \Leftrightarrow (3)$ now follows from $\widetilde{\phi_{\mathscr{U}}} = \bigvee_{\gamma \in \mathrm{Ch}(\mathscr{U})} \bigwedge \mathrm{Im}\,\gamma$. $\square$

Definition 23. Let X be a set, and let $\mathscr{U}, \mathscr{V} \in \mathscr{F}\mathscr{F}X$. Then we say $\mathscr{U}$ is *diagonal* to $\mathscr{V}$, $\mathscr{U} \bowtie \mathscr{V}$, iff $\mathscr{U}$ and $\mathscr{V}$ satisfy any of the equivalent conditions given in Proposition 22.

If we ignore the differences between choice functions and choices, we find that in [13] the diagonality relation is defined in the form of condition (3) above as “the side condition on Cut^*”, and is proved equivalent to condition (5). We have adopted the term “diagonal” from [14], where a pair $(\mathscr{U}, \mathscr{V})$ satisfying the classical version of condition (5) is called a *diagonal pair*.

Proposition 24. (1) *Diagonality is symmetric.*

(2) $\mathscr{U} \bowtie \emptyset \Leftrightarrow \emptyset \in \mathscr{U} \Leftrightarrow \forall \mathscr{V}.\ \mathscr{U} \bowtie \mathscr{V}$.

(3) *If* $W \in \mathscr{F}X$ *then* $\{W\} \bowtie \{\{w\} \mid w \in W\}$.

(4) *If* $\mathscr{W} \in \mathscr{F}\mathscr{F}X$ *then* $\mathscr{W} \bowtie \{\mathrm{Im}\,\gamma \mid \gamma \in \mathrm{Ch}(\mathscr{W})\}$.

(5) *If* $\mathscr{U} \bowtie \mathscr{V}$ *then* $(\mathscr{U} \cup \mathscr{U}') \bowtie (\mathscr{V} \cup \mathscr{V}')$ *for all* $\mathscr{U}'$, $\mathscr{V}'$.

(6) *If* $\mathscr{U}_i \bowtie \mathscr{V}_i$ $(i = 1, 2)$ *then* $(\mathscr{U}_1 \cup \mathscr{U}_2) \bowtie \{V_1 \cup V_2 \mid V_1 \in \mathscr{V}_1, V_2 \in \mathscr{V}_2\}$.

Proof. (1) is obvious.

(2): Suppose $\mathscr{U} \bowtie \emptyset$. $\emptyset$ has a unique choice, with empty image, so $\emptyset \in \mathscr{U}$. If $\emptyset \in \mathscr{U}$ then $\mathscr{U}$ has no choices and so $\mathscr{U} \bowtie \mathscr{V}$ holds vacuously for every $\mathscr{V}$; and when this holds, then in particular we have $\mathscr{U} \bowtie \emptyset$.

(3): The only choice for $\{\{w\} \mid w \in W\}$ has image W.

(4): Obvious (consider choices for $\mathscr{W}$).

(5): For any choice γ' for $\mathscr{U} \cup \mathscr{U}'$ there is a choice γ for $\mathscr{U}$ with $\mathrm{Im}\,\gamma \subseteq \mathrm{Im}\,\gamma'$. Similarly for $\mathscr{V} \cup \mathscr{V}'$. From this we can deduce that any choices for $\mathscr{U} \cup \mathscr{U}'$ and $\mathscr{V} \cup \mathscr{V}'$ have images that meet.

(6): Let γ be a choice for $\mathscr{U}_1 \cup \mathscr{U}_2$. Using Proposition 15 we can then find choices γ_i for $\mathscr{U}_i$ such that $\mathrm{Im}\,\gamma_i \subseteq \mathrm{Im}\,\gamma$, and elements V_i of $\mathscr{V}_i$ such that $V_i \subseteq \mathrm{Im}\,\gamma_i$. Then $V_1 \cup V_2 \subseteq \mathrm{Im}\,\gamma$. □

5. The entailment category

We now present an "information system theoretic" representation **Ent** of $\mathbf{FreeFr}_U^{\mathrm{op}}$.

For an information system for the free frame $\mathbf{Fr}\langle X\rangle$, we take the tokens to be simply the generators X. What we now show is how to express the rest of the structure in terms of the tokens.

The central question is how to describe the morphisms without accepting the frame $\mathbf{Fr}\langle X\rangle$ as a concretely given set. It is impredicative, in that its construction requires the use of powersets. It is also non-geometric, in that it is not preserved by inverse image functors between toposes (again, essentially, because its construction uses powersets and the powerset construction is non-geometric). If the sets X are the information systems, then this central question is (modulo the fact that our morphisms are upper relations, not necessarily continuous maps) analogous to that of describing the *approximable mappings* between information systems.

Once we know what the approximable mappings are, we also have to describe the category structure—identities and composition. This turns out to be remarkably intricate, but in a way that could be anticipated from entailment relations. It is analogous to the problem of deducing $U \vdash W$ (i.e. $\bigwedge U \Rightarrow \bigvee W$) from sequents of the form $U \vdash V$ and $V' \vdash W$, and is the *cut composition* referred to earlier.

Proposition 25. *Let X and Y be sets. Then there is a bijection between*

(1) *preframe homomorphisms* $f : \mathbf{Fr}\langle Y\rangle \to \mathbf{Fr}\langle X\rangle$; *and*

(2) *relations R from $\mathscr{F}X$ to $\mathscr{F}Y$ that are upper closed* (*i.e. if URV then* $(U \cup U')R(V \cup V')$; *we also say that R has weakening*).

Given f, R is defined by

$$URV \quad \textit{iff} \quad \bigwedge U \leqslant f(\bigvee V).$$

Given R,

$$f(b) = \bigvee^{\uparrow} \{\phi_{\mathscr{U}} \mid \exists \mathscr{V}. \ (\mathscr{U}\bar{R}\mathscr{V} \ \textit{and} \ \tilde{\phi}_{\mathscr{V}} \leqslant b)\}$$

where

$$\mathscr{U}\bar{R}\mathscr{V} \quad \textit{iff} \quad \forall U \in \mathscr{U}. \ \forall V \in \mathscr{V}. \ URV.$$

(*In Proposition* 29 *we shall simplify this expression for f.*)

Proof. Because $\mathscr{F}Y$ is the free semilattice over Y, we have

$$\mathbf{Fr}\langle Y\rangle \cong \mathbf{Fr}\langle \mathscr{F}Y \ (\text{qua} \cup = \vee\text{-semilattice})\rangle.$$

The preframe coverage theorem [9] tells us that this is isomorphic to

$$\mathbf{PreFr}\langle \mathscr{F}Y \ (\text{qua poset under } \subseteq)\rangle.$$

Hence a preframe homomorphism $\mathbf{Fr}\langle Y\rangle \to \mathbf{Fr}\langle X\rangle$ is equivalent to a monotone function $\mathscr{F}Y \to \mathbf{Fr}\langle X\rangle$.

On the other hand we can also use the suplattice coverage theorem [2] to deduce

$$\begin{aligned}\mathbf{Fr}\langle X\rangle &\cong \mathbf{Fr}\langle \mathscr{F}X \ (\text{qua} \cup = \wedge\text{-semilattice})\rangle \\ &\cong \mathbf{SupLat}\langle \mathscr{F}X \ (\text{qua poset under } \supseteq)\rangle.\end{aligned}$$

In [6] this is stated concretely so that we see $\mathbf{Fr}\langle X\rangle$ as the set of upper closed subsets of $\mathscr{F}X$.

We have thus given here two distinct representations of free frames, related by distributive lattice duality and by the fact that $\mathbf{Fr}\langle X\rangle$ is the ideal completion $\mathrm{Idl}(\mathbf{DL}\langle X\rangle)$. On the one hand, using conjunctive form of elements of $\mathbf{DL}\langle X\rangle$, we see that every element of $\mathbf{Fr}\langle X\rangle$ is a directed join of finite meets of finite joins of generators. This is the form that was used for defining preframe homomorphisms out of $\mathbf{Fr}\langle Y\rangle$. On the other hand, using disjunctive form, every element of $\mathbf{Fr}\langle X\rangle$ is a join of finite meets of generators. This underlies our concrete representation by the upper closed subsets of $\mathscr{F}X$.

Putting these together, we see that monotone functions $\mathscr{F}Y \to \mathbf{Fr}\langle X\rangle$ correspond to relations R from $\mathscr{F}Y$ to $\mathscr{F}X$ such that each $R(V)$ is upper closed, and if $V' \supseteq V$ then $R(V') \supseteq R(V)$. These are just saying that R has weakening. We shall actually take the relational converse, as in the statement of the Proposition (to get a relation from $\mathscr{F}X$ to $\mathscr{F}Y$), thus giving a localic direction for the morphisms. This direction also matches better that of the sequent calculus, for by following through the technicalities we find that URV iff $\bigwedge U \leqslant f(\bigvee V)$.

Given R, we have

$$f(\bigvee V) = \bigvee\{\bigwedge U \mid URV\} = \bigvee^{\uparrow}\{\phi_{\mathscr{U}} \mid \mathscr{U}\bar{R}\{V\}\}.$$

Because f preserves finite meets, it follows by induction on $\mathscr{V}$ that

$$f(\tilde{\phi}_{\mathscr{V}}) = \bigvee^{\uparrow}\{\phi_{\mathscr{U}} \mid \mathscr{U}\bar{R}\mathscr{V}\}$$

and so

$$f(b) = \bigvee^{\uparrow}\{\phi_{\mathscr{U}} \mid \exists\mathscr{V}.\ (\mathscr{U}\bar{R}\mathscr{V} \text{ and } \tilde{\phi}_{\mathscr{V}} \leqslant b)\}.$$

To see that this join is directed, suppose $\mathscr{U}_i\bar{R}\mathscr{V}_i$ and $\tilde{\phi}_{\mathscr{V}_i} \leqslant b$ $(i = 1, 2)$. Let $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$ and $\mathscr{V} = \{V_1 \cup V_2 \mid V_1 \in \mathscr{V}_1, V_2 \in \mathscr{V}_2\}$. Then $\mathscr{U}\bar{R}\mathscr{V}$ and

$$\begin{aligned}
\phi_{\mathscr{U}} &= \phi_{\mathscr{U}_1} \vee \phi_{\mathscr{U}_2}\\
\tilde{\phi}_{\mathscr{V}} &= \bigwedge\{\bigvee V_1 \vee \bigvee V_2 \mid V_1 \in \mathscr{V}_1, V_2 \in \mathscr{V}_2\}\\
&= \bigwedge_{V_1 \in \mathscr{V}_1} \bigvee V_1 \vee \bigwedge_{V_2 \in \mathscr{V}_2} \bigvee V_2 = \tilde{\phi}_{\mathscr{V}_1} \vee \tilde{\phi}_{\mathscr{V}_2} \leqslant b. \qquad \square
\end{aligned}$$

Proposition 26. *Let X, Y and Z be sets, and let R and S be upper closed relations from $\mathscr{F}X$ to $\mathscr{F}Y$ and from $\mathscr{F}Y$ to $\mathscr{F}Z$ respectively, corresponding to preframe homomorphisms $f : \mathbf{Fr}\langle Y\rangle \to \mathbf{Fr}\langle X\rangle$ and $g : \mathbf{Fr}\langle Z\rangle \to \mathbf{Fr}\langle Y\rangle$. Then the composite $f \circ g$ corresponds to $R \dagger S$ defined by*

$$U(R \dagger S)W \quad \textit{iff} \quad \{U\}(\bar{R}; \bowtie; \bar{S})\{W\}.$$

The identity morphism at X is represented by the relation $\between_X$ ("meets"), defined by $U \between_X V$ iff U meets V.

Proof. We have

$$\begin{aligned}
g(\bigvee W) &= \bigvee^{\uparrow}\{\phi_{\mathscr{V}_2} \mid \mathscr{V}_2\bar{S}\{W\}\}\\
f \circ g(\bigvee W) &= \bigvee^{\uparrow}\{\phi_{\mathscr{U}} \mid \exists\mathscr{V}_1, \mathscr{V}_2.\ \mathscr{U}\bar{R}\mathscr{V}_1, \tilde{\phi}_{\mathscr{V}_1} \leqslant \phi_{\mathscr{V}_2}, \mathscr{V}_2\bar{S}\{W\}\}.
\end{aligned}$$

Now $\bigwedge U \leqslant \phi_{\mathscr{U}}$ iff $U \supseteq U'$ for some $U' \in \mathscr{U}$. So, using Proposition 22,

$$\bigwedge U \leqslant f \circ g(\bigvee W) \quad \text{iff} \quad \exists\mathscr{V}_1, \mathscr{V}_2.\ \{U\}\bar{R}\mathscr{V}_1 \bowtie \mathscr{V}_2\bar{S}\{W\}.$$

The identity morphism comes from our knowledge that $\bigwedge U \leqslant \bigvee V$ in $\mathbf{DL}\langle X\rangle$ iff U meets V. $\square$

Definition 27. The *entailment category* **Ent** is defined as follows.

An object is a set, X.

An *entailment* morphism from X to Y is an upper closed relation R from $\mathscr{F}X$ to $\mathscr{F}Y$.

The identity morphism at X is "meets", $\between_X$.

If $R: X \to Y$ and $S: Y \to Z$ then their composite, the *cut composition*, is $R \dagger S$. (This is a geometric version of the cut composition of [10].)

We have thus shown:

Theorem 28. $\mathbf{FreeFr}_U$ *is dual equivalent to* **Ent**.

We can now simplify the expression for f in Proposition 25. The elements of $\mathbf{Fr}\langle X\rangle$ correspond to the upper closed subsets of $\mathscr{F}X$, and hence to entailment morphisms $X \to \emptyset$.

Proposition 29. *Let* $R: X \to Y$ *be an entailment morphism corresponding to a preframe homomorphism* $f: \mathbf{Fr}\langle Y\rangle \to \mathbf{Fr}\langle X\rangle$. *If* $b \in \mathbf{Fr}\langle Y\rangle$, *then*

$$f(b) = R \dagger b.$$

Proof. We already have an expression for $f(b)$ in Proposition 25. Using it and compactness of $\bigwedge U$, we see that

$$\begin{aligned} U \in f(b) &\Leftrightarrow \bigwedge U \leqslant f(b) \\ &\Leftrightarrow \exists \mathscr{U}, \mathscr{V}.\ \left(\bigwedge U \leqslant \phi_{\mathscr{U}} \text{ and } \mathscr{U}\bar{R}\mathscr{V} \text{ and } \widetilde{\phi_{\mathscr{V}}} \leqslant b\right). \end{aligned}$$

Given such $\mathscr{U}$ and $\mathscr{V}$, this happens if $U \supseteq U'$ for some $U' \in \mathscr{U}$, and there is some $\mathscr{V}'$ with $\mathscr{V} \bowtie \mathscr{V}' \subseteq b$. Hence

$$U \in f(b) \quad \Leftrightarrow \quad U(R \dagger b)\emptyset. \qquad \square$$

If $R: X \to Y$ then its relational converse R° is a morphism from Y to X. It follows from symmetry of $\bowtie$ that $(R \dagger S)^\circ = S^\circ \dagger R^\circ$, and so $^\circ$ is an involution on **Ent**.

Lemma 30. *Suppose in* **Ent** *we have* $R: X \to Y$ *and* $S: Y \to Z$. *Then the following are equivalent*.
(1) $U(R \dagger S)W$.
(2) *There is some* $\mathscr{V}_1 \in \mathscr{F}\mathscr{F}Y$ *such that for all* V_1 *in* $\mathscr{V}_1$ *we have* URV_1 *and for all* γ *in* $\mathrm{Ch}(\mathscr{V}_1)$ *we have* $\mathrm{Im}\,\gamma SW$.
(3) *There is some* $\mathscr{V}_2 \in \mathscr{F}\mathscr{F}Y$ *such that for all* V_2 *in* $\mathscr{V}_2$ *we have* V_2SW *and for all* γ *in* $\mathrm{Ch}(\mathscr{V}_2)$ *we have* $UR\,\mathrm{Im}\,\gamma$.

Proof. $(1 \Rightarrow 2)$: Suppose $\mathscr{V}_1 \bowtie \mathscr{V}_2$ are given as in the definition. If γ is a choice for $\mathscr{V}_1$ then by definition of $\bowtie$ we have that its image contains some V_2 in $\mathscr{V}_2$, and V_2SW.

$(2 \Rightarrow 1)$: Define $\mathscr{V}_2 = \{\mathrm{Im}\,\gamma \mid \gamma \in \mathrm{Ch}(\mathscr{V}_1)\}$.

$(1 \Leftrightarrow 3)$ now follows by symmetry. $\square$

We have argued that **Ent** is a category by showing its equivalence with $\mathbf{SCFr}_U$. However, this uses the impredicative, non-geometric constructions of the frames, so we shall outline a direct proof that **Ent** is a category. From the point of view of geometric mathematics this is not so vital. The categorical structure (composition and so forth) of **Ent** is all geometric and so preserved under inverse image functors. If the properties (associativity and so forth) are known to hold in every topos, then it does not matter if they were proved by non-geometric means using frames. On the other hand for predicative mathematics the frames do not even exist as sets.

By definition we know that

$$U(R \dagger S)W \quad \text{iff} \quad \{U\}(\bar{R}; \bowtie; \bar{S})\{W\},$$

where $\bar{R}$ and $\bar{S}$ are defined as in Proposition 25 and “;” is relational composition. We now generalize this.

Proposition 31. *Let $R : X \to Y$ and $S : Y \to Z$ be entailment morphisms. Then*

$$\overline{R \dagger S} = \bar{R}; \bowtie; \bar{S}.$$

Proof. $\supseteq$ is easy. For $\subseteq$, we first prove by induction on $\mathscr{W}$ that if $\forall W \in \mathscr{W}.\ \mathscr{U}(\bar{R}; \bowtie; \bar{S})\{W\}$ then $\mathscr{U}(\bar{R}; \bowtie; \bar{S})\mathscr{W}$. The base case $\mathscr{W} = \emptyset$ follows from

$$\mathscr{U}\bar{R}\emptyset \bowtie \{\emptyset\}\bar{S}\emptyset.$$

For the induction step, suppose $\mathscr{W} = \mathscr{W}' \cup \{W\}$. Using the induction hypothesis, we can find $\mathscr{V}_1'$, $\mathscr{V}_2'$, $\mathscr{V}_1''$ and $\mathscr{V}_2''$ such that

$$\mathscr{U}\bar{R}\mathscr{V}_1' \bowtie \mathscr{V}_2'\bar{S}\mathscr{W}',$$
$$\mathscr{U}\bar{R}\mathscr{V}_1'' \bowtie \mathscr{V}_2''\bar{S}\{W\}.$$

If we now define $\mathscr{V}_1 = \mathscr{V}_1' \cup \mathscr{V}_1''$ and $\mathscr{V}_2 = \{V' \cup V'' \mid V' \in \mathscr{V}_2',\ V'' \in \mathscr{V}_2''\}$, then by Proposition 24

$$\mathscr{U}\bar{R}\mathscr{V}_1 \bowtie \mathscr{V}_2\bar{S}\mathscr{W}.$$

Returning to the main result, suppose $\mathscr{U}\overline{R \dagger S}\mathscr{W}$. If $U \in \mathscr{U}$ then for every $W \in \mathscr{W}$ we have $\{U\}(\bar{R}; \bowtie; \bar{S})\{W\}$, and so by the lemma $\{U\}(\bar{R}; \bowtie; \bar{S})\mathscr{W}$. Now by the dual of the lemma (applying $^\circ$) we can deduce that $\mathscr{U}(\bar{R}; \bowtie; \bar{S})\mathscr{W}$. $\square$

Proposition 32. **Ent** *is a category; moreover, it is preframe enriched and has involution.*

Proof. The involution, $^\circ$, has already been noted and in fact the duality it provides will be used in the proof here.

From Propositions 21 and 22 we deduce that $\phi_{\mathscr{U}} \leqslant \phi_{\mathscr{V}}$ iff $\mathscr{U}(\bar{\between}; \bowtie)\mathscr{V}$, and the unit laws follow from this.

For associativity,

$$\overline{(Q \dagger R) \dagger S} = \overline{(Q \dagger R)}; \bowtie; \bar{S} = \bar{Q}; \bowtie; \bar{R}; \bowtie; \bar{S} = \bar{Q}; \bowtie; \overline{R \dagger S} = \overline{Q \dagger (R \dagger S)}$$

For the preframe enrichment, each homset is a completely distributive lattice under the set-theoretic operations. What we show is that cut composition distributes over finite intersections and directed unions. In both cases, we need only prove distributivity on the left—the other side follows by duality.

Suppose $R : X \to Y$, $S_i : Y \to Z$ and $U(R \dagger S_i)W$ $(i = 1,2)$ with $\{U\}\bar{R}\mathscr{V}_i \bowtie \mathscr{V}_i{}'\overline{S_i}\{W\}$. Taking $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2$ and $\mathscr{V}' = \{V_1' \cup V_2' \mid V_i' \in \mathscr{V}_i{}'\}$ and using Proposition 24, we see

$$\{U\}\bar{R}\mathscr{V} \bowtie \mathscr{V}'\overline{(S_1 \cap S_2)}\{W\}.$$

For the nullary case, showing $U(R \dagger (\mathscr{F}Y \times \mathscr{F}Z))W$ for all U and W, use $\emptyset \bowtie \{\emptyset\}$. Note that the argument does not extend to infinitary intersections.

Now suppose $(S_i)_{i \in I}$ is a directed family of morphisms from Y to Z. Suppose $U(R \dagger \bigcup_i S_i)W$ with $\{U\}\bar{R}\mathscr{V}_1 \bowtie \mathscr{V}_2\overline{\bigcup_i S_i}\{W\}$. By directedness we can find i with $\mathscr{V}_2\overline{S_i}\{W\}$, and then $U(R \dagger S_i)W$. □

5.1. The cut calculus for products

If $X_1, \ldots, X_n$ are sets then $\mathbf{Fr}\langle X_1\rangle \otimes \cdots \otimes \mathbf{Fr}\langle X_n\rangle$ is a frame coproduct and hence isomorphic to $\mathbf{Fr}\langle X_1 + \cdots + X_n\rangle$ where $+$ denotes disjoint union. Hence a preframe homomorphism $\mathbf{Fr}\langle Y_1\rangle \otimes \cdots \otimes \mathbf{Fr}\langle Y_m\rangle \to \mathbf{Fr}\langle X_1\rangle \otimes \cdots \otimes \mathbf{Fr}\langle X_n\rangle$ is equivalent to an entailment morphism from $X_1 + \cdots + X_n$ to $Y_1 + \cdots + Y_m$, an upper closed relation from $\mathscr{F}(X_1 + \cdots + X_n)$ to $\mathscr{F}(Y_1 + \cdots + Y_m)$. But

$$\mathscr{F}(X_1 + \cdots + X_n) \cong \mathscr{F}X_1 \times \cdots \times \mathscr{F}X_n$$

and so such an entailment morphism is equivalent to an upper closed $m+n$-ary relation on $\mathscr{F}X_1 \times \cdots \times \mathscr{F}X_n \times \mathscr{F}Y_1 \times \cdots \times \mathscr{F}Y_m$. Described in this way, there is no distinction between the domain sets ("input ports") X_i and the codomain sets ("output ports") Y_j. Indeed, one can envisage a calculus of upper closed relations that is neutral as between input and output and later (Section 7) we shall see this in relation to the duality of stably locally compact locales. Meanwhile in this section we shall investigate how cut works across multiple ports. We shall see cut composition generalized so that in many useful situations it can be carried out "port by port" instead of having to be either on all the input ports simultaneously or on all the output ports.

Proposition 33. *Let preframe homomorphisms $f_i : \mathbf{Fr}\langle Y_i\rangle \to \mathbf{Fr}\langle X_i\rangle$ correspond to entailment morphisms $R_i : X_i \to Y_i$ $(i = 1,2)$. Then*

$$f_1 \otimes f_2 : \mathbf{Fr}\langle Y_1\rangle \otimes \mathbf{Fr}\langle Y_2\rangle \to \mathbf{Fr}\langle X_1\rangle \otimes \mathbf{Fr}\langle X_2\rangle$$

corresponds to the entailment morphism $R_1 + R_2 : X_1 + X_2 \to Y_1 + Y_2$ given by

$$U_1 + U_2(R_1 + R_2)V_1 + V_2 \quad \text{iff} \quad U_1R_1V_1 \text{ or } U_2R_2V_2.$$

Proof. Since (with notation as in Section 2) $\bigwedge(U_1 + U_2) = \bigwedge U_1 \times \bigwedge U_2$ and $\bigvee(V_1 + V_2) = \bigvee V_1 \odot \bigvee V_2$, we require

$$(R_1 + R_2)(U_1 + U_2, V_1 + V_2) \;\Leftrightarrow\; \bigwedge U_1 \times \bigwedge U_2 \leqslant f_1(\bigvee V_1) \odot f_2(\bigvee V_2).$$

Using the fact that each $\bigwedge U_i$ is compact (i.e. $\bigwedge U_i \ll \bigwedge U_i$), we can apply Lemma 1 to see that this holds iff $\bigwedge U_1 \leqslant f_1(\bigvee V_1)$ or $\bigwedge U_2 \leqslant f_2(\bigvee V_2)$, from which we deduce the result. $\square$

Lemma 34. *Let $P: X_1 \to X_2$ and $R: X_2 + Y \to Z$ be entailment morphisms. Then $(U_1 + V)((P + \between) \dagger R)W$ holds iff*

$$\exists \mathscr{U}_2 \bowtie \mathscr{U}_2' \text{ in } \mathscr{F}\mathscr{F}X_2 \text{ such that } \{U_1\}\bar{P}\mathscr{U}_2 \text{ and } \forall U_2' \in \mathscr{U}_2'.\ (U_2' + V)RW.$$

Proof. $\Rightarrow$: Suppose $(U_1 + V)((P + \between) \dagger R)W$. Then we have $\mathscr{T}$ and $\mathscr{T}'$ in $\mathscr{F}(\mathscr{F}X_2 \times \mathscr{F}Y)$ with the conditions for cut composition. For every $U_2 + V'$ in $\mathscr{T}$ (U_2 in $\mathscr{F}X_2$, V' in $\mathscr{F}Y$) we have either $U_1 P U_2$ or $V \between V'$, so we can decompose $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ such that for every $U_2 + V'$ in $\mathscr{T}_1$ we have $U_1 P U_2$ and for every $U_2 + V'$ in $\mathscr{T}_2$ we have $V \between V'$. If $\mathscr{V} = \{V' \mid U_2 + V' \in \mathscr{T}_2\}$, then it follows that there is a choice δ of $\mathscr{V}$ whose image is included in V.

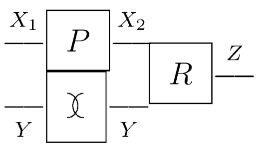
Let $\mathscr{U}_2 = \{U_2 \mid U_2 + V' \text{ in } \mathscr{T}_1\}$ and let $\mathscr{U}_2' = \{\operatorname{Im}\gamma \mid \gamma \in \operatorname{Ch}(\mathscr{U}_2)\}$. Suppose $\gamma \in \operatorname{Ch}(\mathscr{U}_2)$. Then γ and δ together provide a choice of $\mathscr{T}$, and it follows that $(\operatorname{Im}\gamma + \operatorname{Im}\delta)RW$ and so $(\operatorname{Im}\gamma + V)RW$ as required.

$\Leftarrow$: Suppose we have $\mathscr{U}_2$ and $\mathscr{U}_2'$ as stipulated. Let

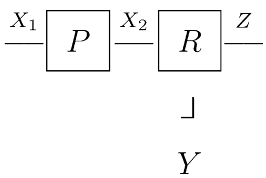
$$\mathscr{T} = \{U_2 + \emptyset \mid U_2 \in \mathscr{U}_2\} \cup \{\emptyset + \{v\} \mid v \in V\}.$$

Then for every $U_2 + V'$ in $\mathscr{T}$ we have either $U_1 P U_2$ or $V \between V'$. Any choice of $\mathscr{T}$ is got from a choice γ of $\mathscr{U}_2$ and the choice of every $v \in V$, so its image is $\operatorname{Im}\gamma + V$. Since $\operatorname{Im}\gamma$ includes some $U_2' \in \mathscr{U}_2'$, we deduce that $(\operatorname{Im}\gamma + V)RW$. $\square$

The lemma tells how to simplify cut composition in certain circumstances, applying it at one port at a time. In the lemma, we can apply it at just X_2. Pictorially,

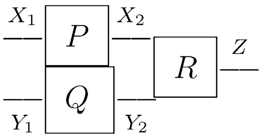


(cutting X_2 and Y together) is the same as



cutting just on X_2.

Thus we ignore the Y port when we calculate the cut composition. This is only natural, since the entailment morphism on that port is just the identity. More generally, this observation allows us to calculate cut compositions in steps, as follows. Suppose we have $P : X_1 \to X_2$, $Q : Y_1 \to Y_2$ and $R : X_2 + Y_2 \to Z$, and we desire to cut at X_2 and Y_2 together:



Then $P+Q = (P + \between) \dagger (\between + Q)$—this is obvious by considering composition of preframe homomorphisms, or alternatively can be proved by elementary means after applying the lemma. We deduce that $(P+Q) \dagger R$ can be got by applying cut composition of Q with R at Y_2 (ignoring X_2), and then cutting P with the result at X_2 (ignoring Y_1). We can denote this by

$$P \dagger_{X_2} (Q \dagger_{Y_2} R).$$

Of course, we get the same result if we cut first at X_2 and then at Y_2.

We have considered only the situation where P and Q are separate. We leave for further study the more general situation, e.g. with $P : X \to Y_1 + Y_2$ and $Q : Y_1 + Y_2 \to Z$.

5.2. *Some spatial aspects of free frames*

A point of the locale corresponding to $\mathrm{Fr}\langle X \rangle$ is just a subset of X, in other words an ideal of $\mathscr{F}X$, and it follows that $\mathrm{Fr}\langle X \rangle$ is the frame of the lower powerlocale $P_L X \cong \mathbb{S}^X$ (where $\mathbb{S}$ is the Sierpiński locale [22]). To put it another way, $\mathrm{Fr}\langle X \rangle$ is the Scott topology on the powerset $\mathscr{P}X$.

$$\mathrm{Idl}(\mathrm{DL}\langle X \rangle) \cong \mathrm{Fr}\langle X \rangle \cong \mathrm{Alex}(\mathscr{F}X) \cong \mathrm{Scott}(\mathscr{P}X) \cong \Omega\mathbb{S}^X \cong \Omega P_L X$$

(Alex is the Alexandrov topology, Scott the Scott topology.)

Definition 35. Define $\bowtie$ as a relation on $\mathrm{DL}\langle X \rangle$ by $\phi \bowtie \psi$ iff $\phi \geqslant \tilde{\psi}$ (cf. Proposition 22).

As a predicate, a function from $\mathrm{DL}\langle X \rangle \times \mathrm{DL}\langle X \rangle$ to Ω, note that $\bowtie$ is a bimorphism with respect to $\wedge$:

$$\phi \bowtie \bigwedge_i \psi_i \quad \Leftrightarrow \quad \phi \geqslant \bigvee_i \widetilde{\psi_i} \quad \Leftrightarrow \quad \forall i.\ \phi \geqslant \widetilde{\psi_i} \quad \Leftrightarrow \quad \forall i.\ \phi \bowtie \psi_i$$

We can now extend $\bowtie$ to a relation on $\mathrm{Idl}(\mathrm{DL}\langle X \rangle) = \mathrm{Fr}\langle X \rangle$ by $a \bowtie b$ iff $\phi \bowtie \psi$ for some ϕ, ψ in $\mathrm{DL}\langle X \rangle$ with $\phi \leqslant a$ and $\psi \leqslant b$. This then gives a preframe bimorphism $\Omega\mathbb{S}^X \times \Omega\mathbb{S}^X \to \Omega$ and hence a preframe homomorphism $\bowtie\, : \Omega(\mathbb{S}^X \times \mathbb{S}^X) \cong \Omega\mathbb{S}^X \otimes \Omega\mathbb{S}^X \to \Omega$, $\bowtie (a \odot b) = \mathbf{true}$ iff $a \bowtie b$. By the Hofmann–Mislove–Johnstone theorem (see [22]) this then corresponds to a compact fitted sublocale K of $\mathbb{S}^X \times \mathbb{S}^X$, namely the meet of all the open sublocales $a \odot b$ for which $a \bowtie b$.

Proposition 36. *The points of this sublocale are the pairs of subsets (A, B) such that $A \cup B = X$.*

Proof. Let F be the corresponding Scott open filter of $\Omega(\mathbb{S}^X \times \mathbb{S}^X)$. (A,B) is in K iff it is in every $u \in F$, and it suffices to test for those u's of the form $\bigvee U \odot \bigvee V$, for they give a preframe basis. Now $\bigvee U \bowtie \bigvee V$ iff $\bigvee U \geqslant \bigwedge V$, i.e. iff U meets V. Hence (A,B) is in K iff for every $U, V \in \mathscr{F}X$, if U meets V then either U meets A or V meets B, and this holds iff $A \cup B = X$. For if $x \in X$ then $\{x\}$ meets itself, so x is in either A or B; while conversely if U meets V in x then x is in either A or B and so either U meets A in x or V meets B in x. □

Corollary 37 (Classically). *$\{(A,B) \mid A \cup B = X\}$ is a compact subspace of $\mathscr{P}X \times \mathscr{P}X$.*

Proof. Use the Hofmann–Mislove theorem. □

We can also analyse the cut-composition in these terms. For an entailment morphism $R: X \to Y$ is an Alexandrov open in $\mathscr{F}X \times \mathscr{F}Y \cong \mathscr{F}(X+Y)$, and hence a Scott open in $\mathscr{P}X \times \mathscr{P}Y \cong \mathscr{P}(X+Y)$.

Proposition 38. *Suppose $R: X \to Y$ and $S: Y \to Z$ in* **Ent**. *Then $(A,C) \models R \dagger S \Leftrightarrow \forall B_1 \cup B_2 = Y.\ ((A,B_1) \models R \vee (B_2, C) \models S)$.*

(*Following* [20], *we write $x \models a$ when a point x satisfies an open a.*)

Proof. $\Rightarrow$: We can find $A_0 \in \mathscr{F}A$ and $C_0 \in \mathscr{F}C$ with $A_0(R \dagger S)C_0$, and then $\{A_0\}\bar{R}\mathscr{V}_1 \bowtie \mathscr{V}_2\bar{S}\{C_0\}$. If $B_1 \cup B_2 = Y$ then $(B_1, B_2) \models \phi_{\mathscr{V}_1} \odot \phi_{\mathscr{V}_2}$, so either $V_1 \subseteq B_1$ for some $V_1 \in \mathscr{V}_1$, in which case $(A, B_1) \models R$, or $V_2 \subseteq B_2$ for some $V_2 \in \mathscr{V}_2$, in which case $(B_2, C) \models S$.

$\Leftarrow$: Let b_1 be the inverse image of R along $\langle A, \mathrm{Id}\rangle : \mathscr{P}Y \to \mathscr{P}X \times \mathscr{P}Y$ and let b_2 be the inverse image of S along $\langle \mathrm{Id}, C\rangle : \mathscr{P}Y \to \mathscr{P}Y \times \mathscr{P}Z$. The condition says that $b_1 \odot b_2 \in F$, and so b_1 and b_2 have diagonal finite subsets $\mathscr{V}_1$ and $\mathscr{V}_2$. If $V_1 \in \mathscr{V}_1$ then $V_1 \in b_1 \Rightarrow (A, V_1) \models R$. Hence A has a finite subset A_0 such that $A_0 R V_1$ for every $V_1 \in \mathscr{V}_1$. Similarly, C has a finite subset C_0 such that $V_2 S C_0$ for every $V_2 \in \mathscr{V}_2$. It follows that $A_0(R \dagger S)C_0$ and so $(A,C) \models R \dagger S$. □

Now—at least classically—the unit and associativity laws are less surprising.

6. Entailment systems

We now put together our results to gain a theory of information systems for stably locally compact locales. We have seen that **Ent** is dual to $\mathbf{FreeFr}_U$, and so **Kar**(**Ent**) is dual to **Kar**($\mathbf{FreeFr}_U$), which in turn is equivalent to $\mathbf{SCFr}_U$. It follows that **Kar**(**Ent**) is equivalent to $\mathbf{SLCLoc}_U$.

It is worth remarking that there is an analogous (and simpler) result for continuous dcpos. Let $\mathbf{CtsDcpo}_L$ be the Kleisli category for the lower powerlocale P_L restricted to the continuous dcpos (in localic form). Then [21] shows in effect that $\mathbf{CtsDcpo}_L$ is equivalent to **Kar**(**Rel**), where **Rel** is the category of sets and relations. That is because an idempotent in **Rel** is just a transitive, interpolative relation, in other words what that paper calls an “infosys”. Again, we see locales being obtained by splitting idempotents in a category of structures with the nature of information systems.

Definition 39. An *entailment system* is an object of $\mathbf{Kar}(\mathbf{Ent})$, i.e. a set X equipped with an idempotent endomorphism $\vdash : X \to X$ in **Ent**.

We write $\mathrm{Spec}(X, \vdash)$ for the corresponding stably locally compact locale, and $\Omega(X, \vdash)$ for its stably continuous frame of opens.

We should compare these with the coherent sequent calculi of [10], which are the same as the continuous sequent calculi of [13]. Rather than a simple set X, they use a structure equipped with operations $\wedge$, $\vee$, $\top$ and $\perp$ corresponding to the lattice operations (but with no laws assumed). However, [13] shows how relations on $\mathscr{F}X$ can be extended to relations on the finite powerset of the term algebra generated by X and those operations. A more significant difference is their assumption of *interpolants* as mentioned earlier. With this, the idempotence $\vdash = \vdash \dagger \vdash$ (their Cut^* rule) is equivalent to a simpler rule Cut':

$$\frac{S \vdash T \cup \{\phi\} \qquad \{\phi\} \cup U \vdash V}{S \cup U \vdash T \cup V}.$$

The reliance on a single ϕ here needs the ability to form meets and joins of tokens. But even when this is done freely, our lack of interpolants means that Cut' is not in general valid for our entailment systems.

Proposition 40. *Let* $(X, \vdash)$ *be an entailment system and let* $j : \mathbf{Fr}\langle X \rangle \to \mathbf{Fr}\langle X \rangle$ *be the corresponding preframe endomorphism..*

(1) *Opens of* $\mathrm{Spec}(X, \vdash)$ *are equivalent to upper closed subsets* a *of* $\mathscr{F}X$ *that, considered as entailment morphisms* $X \to \emptyset$, *satisfy* $a = \vdash \dagger a$.

(2) *Treating elements of* $\mathbf{Fr}\langle X \rangle$ *and* $\Omega(X, \vdash)$ *as entailment morphisms* $X \to \emptyset$,

$$j_{\mathrm{in}}(b) = b,$$
$$j_{\mathrm{out}}(a) = \vdash \dagger a.$$

If $V \in \mathscr{F}X$ *then the open* $j_{\mathrm{out}}(\bigvee V)$ *corresponds to* $\check{V}$ *defined by* $U \check{V} \emptyset$ *iff* $U \vdash V$.

(3) *The opens* $\check{V}$ *form a preframe basis. If* a *is an open then*

$$a = \bigvee^{\uparrow} \left\{ \bigwedge_{V \in \mathscr{V}_1} \check{V} \mid \mathscr{V}_1 \bowtie \mathscr{V}_2 \subseteq a \right\}.$$

Proof. 1. An open of $\mathrm{Spec}(X, \vdash)$ is a continuous map from $\mathrm{Spec}(X, \vdash)$ to the Sierpiński locale $\mathbb{S}$. Since $\mathbb{S} \cong P_U 1$ and 1 is the locale for the initial frame Ω (free on no generators), an open of $\mathrm{Spec}(X, \vdash)$ is equivalent to a morphism $(X, \vdash) \to (\emptyset, \emptyset)$ of entailment systems.

2. By considering all as upper closed subsets of $\mathscr{F}X$, we are identifying each $b \in \Omega(X, \vdash)$ with its image $j_{\mathrm{in}}(b)$ in $\mathbf{Fr}\langle X \rangle$. By Proposition 29, $j_{\mathrm{out}}(a) = j(a) = \vdash \dagger a$. U is in the open $j_{\mathrm{out}}(\bigvee V)$ iff $\bigwedge U \leqslant j(\bigvee V)$, i.e. iff $U \vdash V$.

3. Since the elements $\bigvee V$ form a preframe basis of $\mathbf{Fr}\langle X\rangle$, it follows that the elements $j_{\mathrm{out}}(\bigvee V)$ form a preframe basis of $\mathbf{Fr}\langle X\rangle_j$. The join described is directed, for if $\mathscr{V}_1 \bowtie \mathscr{V}_2 \subseteq a$ and $\mathscr{V}_1' \bowtie \mathscr{V}_2' \subseteq a$ then by taking $\mathscr{V}_1'' = \{V \cup V' \mid V \in \mathscr{V}_1, V' \in \mathscr{V}_1'\}$ and $\mathscr{V}_2'' = \mathscr{V}_2 \cup \mathscr{V}_2'$ we get an upper bound for $\bigwedge_{V\in\mathscr{V}_1} \check{V}$ and $\bigwedge_{V\in\mathscr{V}_1'} \check{V}$. Now

$$U \in \bigwedge_{V\in\mathscr{V}_1} \check{V} \quad \text{iff} \quad \{U\}\overline{\vdash}\mathscr{V}_1$$

and

$$\mathscr{V}_2 \subseteq a \quad \text{iff} \quad \mathscr{V}_2 \bar{a} \emptyset,$$

so it follows that U is in the join described iff $U(\vdash \dagger\, a)\emptyset$, i.e. iff $Ua\emptyset$. $\square$

Proposition 41. *Let a morphism $R:(X,\vdash)\to(Y,\vdash)$ of entailment systems correspond to a preframe homomorphism $f:\Omega(Y,\vdash)\to\Omega(X,\vdash)$. Then*

$$f(b) = R \dagger b,$$
$$URV \ \textit{iff} \ U \in f(\check{V}).$$

Proof.

$$f(b) = j_{\mathrm{out}}(R \dagger j_{\mathrm{in}}(b)) = \vdash \dagger R \dagger b = R \dagger b.$$
$$U \in f(\check{V}) \ \Leftrightarrow \ U(R \dagger \check{V})\emptyset \ \Leftrightarrow \ U(R\dagger \vdash)V \ \Leftrightarrow \ URV. \qquad \square$$

Constructions on stably locally compact locales can now be carried out on the entailment systems.

The *product* of entailment systems $(X,\vdash)$ and $(Y,\vdash)$ is (by Proposition 8) $(X,\vdash)\times(Y,\vdash) = (X+Y, \vdash + \vdash)$, with

$$U_1 + V_1(\vdash + \vdash)U_2 + V_2 \quad \text{iff} \quad U_1 \vdash U_2 \ \text{or} \ V_1 \vdash V_2.$$

The projection map $\pi_X:(X,\vdash)\times(Y,\vdash)\to(X,\vdash)$ is defined by

$$(U+W)\pi_X V \quad \text{iff} \quad U \vdash V \ \text{or} \ W \vdash \emptyset$$

and similarly for Y. (Of course, it is not a categorical product in $\mathbf{Kar}(\mathbf{Ent})$ since the morphisms correspond to the upper relations between locales, not the continuous maps.) If A and B are the free frames on X and Y and j and k are the corresponding quasinuclei, then what is required is $(U+W)\pi_X V$ iff $\bigwedge U \times \bigvee V \odot \bigvee \emptyset$. By Lemma 1 this happens iff $\bigwedge U \leqslant \bigvee V$ or $\bigwedge W \leqslant \bigvee \emptyset$, thus giving the result.

We now look at the points of $\mathrm{Spec}(X,\vdash)$. More generally, amongst the upper relations we determine the continuous maps.

Theorem 42. *Let $R:(X,\vdash)\to(Y,\vdash)$ be a Karoubi morphism between two entailment systems. Then the following are equivalent.*
(1) *The corresponding upper relation is a map.*

(2) *$R \dagger \emptyset \subseteq \vdash_X \dagger \emptyset$ and for all $V_1, V_2 \in \mathscr{F}Y$ we have*

$$R \dagger (\check{V}_1 \cup \check{V}_2) \subseteq \vdash \dagger (R \dagger \check{V}_1 \cup R \dagger \check{V}_2).$$

(3) *For every $\mathscr{V} \in \mathscr{F}\mathscr{F}Y$ we have*

$$R \dagger \bigcup_{V \in \mathscr{V}} \check{V} \subseteq \vdash \dagger \bigcup_{V \in \mathscr{V}} (R \dagger \check{V}).$$

Proof. (2) comprises the nullary and binary versions of (3), so they are equivalent. They say that R preserves finite joins of preframe basic opens $\check{V}$, but from that we deduce that it preserves finite joins of arbitrary opens. □

Corollary 43. *A point of* Spec$(X, \vdash)$ *is an upper closed subset P of $\mathscr{F}X$ such that $P \dagger \vdash = P$ (treating P as an entailment morphism from $\emptyset$ to Y), $P \dagger \emptyset = \emptyset$ and for all $U_1, U_2 \in \mathscr{F}X$ we have $P \dagger (\check{U}_1 \cup \check{U}_2) \subseteq P \dagger \check{U}_1 \cup P \dagger \check{U}_2$.*

Proof. Apply the previous theorem, exchanging X and Y and taking $Y = \emptyset$. □

Corollary 44. *We can present $\Omega(X, \vdash)$ as*

$$\begin{aligned}
\mathrm{Fr}\langle \mathscr{F}X\ (\textit{qua poset under } \subseteq) \mid U = {} & \textstyle\bigvee^{\uparrow} \{ \bigwedge \mathscr{V} \mid \mathscr{V} \bowtie \mathscr{V}' \overline{\vdash} \{U\} \} \\
(\emptyset) \leqslant {} & \mathbf{false} \\
\textstyle\bigwedge \mathscr{V} \leqslant {} & (U_1) \vee (U_2) \\
& \qquad (\mathscr{V} \bowtie \mathscr{V}'_1 \cup \mathscr{V}'_2, \mathscr{V}'_i \overline{\vdash} \{U_i\}) \rangle.
\end{aligned}$$

Proof. What we are doing here is making explicit the fact that points (as described before) are the models of a propositional geometric theory.

The condition $P \dagger \vdash = P$ says that $U \in P$ iff there are $\mathscr{V} \bowtie \mathscr{V}' \overline{\vdash} \{U\}$ such that $\{\emptyset\} \bar{P} \mathscr{V}$, which says that the point satisfies $\bigwedge \mathscr{V}$.

The condition $P \dagger \emptyset = \emptyset$ says that it is impossible to have $\{\emptyset\} \bar{P} \mathscr{V} \bowtie \mathscr{V}' \bar{\emptyset} \{\emptyset\}$. But for this to happen we must have $\mathscr{V}' = \emptyset$, so $\emptyset \in \mathscr{V}$, so it is asserting the impossibility of $\emptyset P \emptyset$.

The final condition $P \dagger (\check{U}_1 \cup \check{U}_2) \subseteq P \dagger \check{U}_1 \cup P \dagger \check{U}_2$ must be reworked slightly to get it into the form of the relation. We have $\emptyset (P \dagger (\check{U}_1 \cup \check{U}_2)) \emptyset$ iff we can find $\mathscr{V} \bowtie \mathscr{V}_1' \cup \mathscr{V}_2'$ with $\mathscr{V}_i' \overline{\vdash} \{U_i\}$ and $\mathscr{V} \subseteq P$ (i.e. P is in $\bigwedge \mathscr{V}$). In these circumstances, the relation asserts that $\emptyset P U_i$ for some i, either 1 or 2, and in the presence of the first condition this is equivalent to $\emptyset (P \dagger \vdash) U_i$, i.e. $\emptyset (P \dagger \check{U}_i) \emptyset$. □

6.1. Reflexive entailment relations

We say that an entailment endomorphism R is *reflexive* iff it includes $\between$, i.e. it satisfies reflexivity $\{x\} R \{x\}$.

Proposition 45. *A reflexive entailment endomorphism R is an entailment relation* (*i.e. idempotent for* $\dagger$) *iff it satisfies the cut rule*

$$\frac{UR(\{v\}\cup W) \quad (U\cup\{v\})RW}{URW}.$$

Proof. $\Rightarrow$: Given the premises, we have

$$\{U\}\bar{R}(\{\{u\} \mid u\in U\}\cup\{\{v\}\cup W\}) \\ \bowtie(\{U\cup\{w\} \mid w\in W\}\cup\{U\cup\{v\}\})\bar{R}\{W\}$$

so $U(R\dagger R)W$.

$\Leftarrow$: $R = R\dagger\between\subseteq R\dagger R$ by reflexivity.

By induction on V, one can show that if $(U\cup V)RW$ and $\forall v\in V.\ UR(\{v\}\cup W)$ then URW. Next by induction on $\mathscr{V}$ one can show that if $\forall V\in\mathscr{V}.\ (U\cup V)RW$ and $\forall\delta\in\mathrm{Ch}(\mathscr{V}).\ UR(\mathrm{Im}\,\delta\cup W)$ then URW. This gives the result, for if $\{U\}\bar{R}\mathscr{V}_1\bowtie\mathscr{V}_2\bar{R}\{W\}$ then we can take $\mathscr{V}=\mathscr{V}_2$. $\square$

Reflexivity of an entailment relation is equivalent to the corresponding quasinucleus being inflationary, hence a nucleus. Therefore we have a bijection between reflexive entailment relations on X and perfect (Scott continuous) nuclei on $\mathbf{Fr}\langle X\rangle$. Now a nucleus j on a stably continuous frame is perfect iff j_{out} preserves $\ll$, so on a coherent frame (such as $\mathbf{Fr}\langle X\rangle$), j is perfect iff j_{out} preserves compactness. It follows that there is a bijection between reflexive entailment relations and quotients of $\mathbf{DL}\langle X\rangle$. This has been proved by other means in [CC00].

7. Duality

We now turn to an application that illustrates the power of the entailment calculus.

At the level of presentations, it is plain that there is a duality: if $(X,\vdash)$ is an entailment system, then so is $(X,\dashv)$ where we write $\dashv$ for the relational converse $\vdash^\circ$. Moreover, there is an equivalence between morphisms $(X,\vdash)\times(Y,\vdash)\to(Z,\vdash)$ and morphisms $(X,\vdash)\to(Y,\dashv)\times(Z,\vdash)$. Each is a ternary relation R on $\mathscr{F}X\times\mathscr{F}Y\times\mathscr{F}Z$, in the first case subject to conditions

$$(\vdash+\vdash)\dagger R = R = R\dagger\vdash.$$

However, the remarks following Lemma 34 show that the left hand equation can be split up as $\vdash\dagger_X R = R = \vdash\dagger_Y R$. But the Y part of this is equivalent to $R\dagger_Y\dashv\, = R$, so by similar reasoning we see that the conditions on R needed to give a morphism $(X,\vdash)\times(Y,\vdash)\to(Z,\vdash)$ are the same as those needed to give a morphism $(X,\vdash)\to(Y,\dashv)\times(Z,\vdash)$.

The aim of this section is to show that these simple consequences of the cut calculus are in fact independent of presentation and correspond to the known duality of stably locally compact locales (or spaces) and the monoidal closure of $\mathbf{SCFr}_U$.

In [3,6] there is seen a duality for stably continuous frames and hence for stably locally compact locales. The classical analogue for topological spaces is that if X is stably locally compact then the dual space has the same points but with the *cocompact* topology: the open sets are the complements of the compact saturated subspaces in the original topology. Its specialization order is opposite to the original. For locale theory, if A is a stably continuous frame, then its Hofmann–Lawson dual $\tilde{A}$ is the set of Scott open filters on A (equivalently, the set of preframe homomorphisms $A \to \Omega$). We shall show that this locale duality corresponds to the entailment system duality mentioned above.

The monoidal category **PreFr** is monoidal closed, using the fact that if P and Q are preframes then so is $P \pitchfork Q = \mathbf{PreFr}(P,Q)$. We shall show that $\mathbf{SCFr}_U$ is closed under $\pitchfork$, in fact with $A \pitchfork B \cong \tilde{A} \otimes B$.

Suppose we have three entailment systems $(X,\vdash)$, $(Y,\vdash)$ and $(Z,\vdash)$ and a morphism

$$R : (X,\vdash) \to (Y,\vdash) \times (Z,\vdash).$$

R is a ternary relation on $\mathscr{F}X \times \mathscr{F}Y \times \mathscr{F}Z$. Suppose the corresponding stably continuous frames are A, B and C. Then R gives a preframe homomorphism $B \otimes C \to A$ and hence $C \to B \pitchfork A \cong \tilde{B} \otimes A$, which gives an entailment morphism

$$S : (X,\vdash) \times (Y,\dashv) \to (Z,\vdash).$$

It turns out that, as ternary relation on $\mathscr{F}X \times \mathscr{F}Y \times \mathscr{F}Z$, S is the same as R, thus matching the equivalence described earlier.

Proposition 46. *Let $(X,\vdash)$ be an entailment system. Then there is a preframe isomorphism*

$$\Omega(X,\dashv) \cong \mathbf{PreFr}(\Omega(X,\vdash),\Omega).$$

Proof. An element of $\Omega(X,\dashv)$ is an upper closed subset b of $\mathscr{F}X$ that, considered as an entailment morphism $X \to \emptyset$, satisfies $b = \dashv \dagger b$. But this is equivalent to the condition $b^\circ = b^\circ \dagger \vdash$ on the dual $b^\circ : \emptyset \to X$. These are equivalent to entailment system morphisms $(\emptyset,\emptyset) \to (X,\vdash)$, and hence to preframe homomorphisms $\Omega(X,\vdash) \to \Omega$.

To be explicit, each element b of $\Omega(X,\dashv)$ acts on $\Omega(X,\vdash)$ by $a \mapsto b^\circ \dagger a$. $\square$

It follows that if A is a stably continuous frame then so is $\tilde{A} = A \pitchfork \Omega$, and $\tilde{\tilde{A}} \cong A$. In fact, the natural preframe homomorphism $A \to \tilde{\tilde{A}}$ is an isomorphism.

Proposition 47. *Let $R : (X,\vdash) \to (Y,\vdash)$ be a morphism of entailment systems, and let $f : B \to A$ be the corresponding preframe homomorphism. Then $R^\circ : (Y,\dashv) \to (X,\dashv)$ corresponds to $\tilde{f} : \tilde{A} \to \tilde{B}$.*

Proof. An open of $(X, \dashv)$ is an entailment morphism $c = \dashv \dagger c : X \to \emptyset$, and R° takes it to $R^\circ \dagger c$. This acts as element of $\tilde{B}$ by

$$\begin{aligned} b &\mapsto (R^\circ \dagger c)^\circ \dagger b = c^\circ \dagger R \dagger b \\ &= c^\circ \dagger f(b) = \tilde{f}(c^\circ \dagger \_)(b). \qquad \square \end{aligned}$$

If A is a stably continuous frame, we write $\mathrm{ev} : \tilde{A} \otimes A \to \Omega$ for the evaluation morphism, defined by

$$\mathrm{ev}(F \odot a) = [a \in F],$$

where we use the square brackets to denote the truth value.

Proposition 48. *Let* $(X, \vdash)$ *be an entailment system with* $A = \Omega(X, \vdash)$. *Then* $\mathrm{ev} : \tilde{A} \otimes A \to \Omega$ *corresponds to the morphism of entailment systems* $\varepsilon : \emptyset \to (X, \dashv) \times (X, \vdash)$ *defined by*

$$\emptyset \varepsilon (U + V) \quad \textit{iff} \quad U \vdash V.$$

Proof. The preframe basics for $\tilde{A}$ are (the duals of) the entailment morphisms $\hat{U} : (\emptyset, \emptyset) \to (X, \vdash)$ defined by $\emptyset \hat{U} V$ iff $U \vdash V$. Then

$$\begin{aligned} \mathbf{true} \leqslant \mathrm{ev}(\hat{U} \odot \check{V}) &\Leftrightarrow \emptyset(\hat{U} \dagger \check{V})\emptyset \\ &\Leftrightarrow U(\vdash \dagger \vdash)V \Leftrightarrow U \vdash V. \qquad \square \end{aligned}$$

If $A = \Omega(X, \vdash)$, then let us write $\mathrm{coev} : \Omega \to A \otimes \tilde{A}$ for the preframe homomorphism corresponding to the morphism of entailment systems

$$\delta : (X, \vdash) \times (X, \dashv) \to (\emptyset, \emptyset)$$

defined by $(U + V)\ \delta \emptyset$ iff $U \vdash V$.

(On the face of it, coev depends on the presentation $(X, \vdash)$ of A. However, Proposition 49 will characterize it uniquely with respect to ev, whose definition is not presentation dependent.)

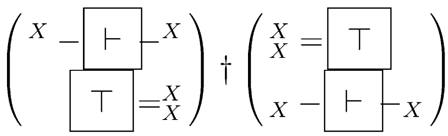
Note that if A is the frame of a stably locally compact locale X then the elements of $\tilde{A}$, the Scott open filters of ΩX, correspond by the Hofmann–Mislove–Johnstone theorem (see [22]) to the compact fitted sublocales of X. A filter F corresponds to the meet $\bigwedge\{a \mid a \in F\}$ of the open sublocales for its elements. Let us write K_F for this compact fitted sublocale. Then $a \in F \Leftrightarrow K_F \leqslant a$, and $F \subseteq \{a' \mid a \ll a'\} \Leftrightarrow a \leqslant K_F$, so $G \ll F$ iff an open sublocale can be interpolated between K_F and K_G. Similarly, $a \ll b$ iff a compact fitted sublocale can be interpolated between a and b. See [3] for more details.

Proposition 49. *Let* A *be a stably continuous frame. Then the composite*

$$(\mathrm{coev} \otimes A); (A \otimes \mathrm{ev}) : A \cong \Omega \otimes A \to A \otimes \tilde{A} \otimes A \to A \otimes \Omega \cong A$$

is the identity.

Proof. This is immediate by the cut calculus (see Lemma 34). The required composition is



and this is equal to $\vdash \dagger \vdash \dagger \vdash \dagger \vdash\, = \vdash$. $\square$

Proposition 50. *Let A be a stably continuous frame. Then the functor* $(-\otimes A)$: **PreFr** $\to$ **PreFr** *is left adjoint to* $(-\otimes \tilde{A})$.

Proof. The unit and counit of the adjunction are given by

$$\cong; (P \otimes \mathrm{coev}) : P \cong P \otimes \Omega \to P \otimes A \otimes \tilde{A}$$

and

$$(Q \otimes \mathrm{ev}); \cong : Q \otimes \tilde{A} \otimes A \to Q \otimes \Omega \cong Q.$$

The diagonal identities are provided by Proposition 49. $\square$

Theorem 51. $\mathbf{SCFr}_U$ *is monoidal closed.*

Proof. Proposition 50 shows that if A is a stably continuous frame then for any preframe C the preframe hom $A \pitchfork C$ is isomorphic to $C \otimes \tilde{A}$, and so if C is also a stably continuous frame then so is $A \pitchfork C$.

The evaluation morphism is

$$(C \otimes \mathrm{ev}); \cong : C \otimes \tilde{A} \otimes A \to C \otimes \Omega \cong C.$$

Given $\alpha : B \otimes A \to C$, the corresponding morphism $B \to C \otimes \tilde{A}$ is

$$\cong; (B \otimes \mathrm{coev}); (\tilde{A} \otimes \alpha) : B \cong B \otimes \Omega \to B \otimes \tilde{A} \otimes A \to C \otimes \tilde{A}.$$

On locales, we denote the duality by $^\circ$: so upper relations $X \times Y \to Z$ are equivalent to upper relations $X \to Z \times Y^\circ$.

We now look at how the entailment calculus represents the duality transpose of Proposition 50. Suppose stably continuous frames A, B and C are presented by entailment systems $(X, \vdash)$, $(Y, \vdash)$ and $(Z, \vdash)$, and a preframe homomorphism θ from B to $A \pitchfork C \cong C \otimes \tilde{A}$ corresponds to a morphism of entailment systems

$$P : (Z, \vdash) \times (X, \dashv) \to (Y, \vdash).$$

This is a ternary relation P on $\mathscr{F}Z \times \mathscr{F}X \times \mathscr{F}Y$ such that $P \dagger_Y \vdash_Y\, = P$ and

$$(\vdash_Z + \dashv_X) \dagger_{Z+X} P = P.$$

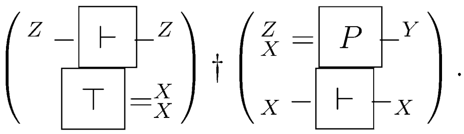
But by the cut calculus this second condition can be simplified to

$$\vdash_Z \dagger_Z P = P$$

and

$$\dashv_X \dagger_X P = P.$$

Now it is clear that so far as the entailment morphisms go, this final condition can equally well be expressed as $P \dagger_X (\dashv_X)^\circ = P$—the cut composition is quite symmetric in its definition, and we just have to be clear about which end of $\vdash$ is being composed. Hence such a ternary relation P is also equivalent to a morphism from $(Z, \vdash)$ to $(Y, \vdash) \times (X, \vdash)$. This equivalence matches the transpose of Proposition 50, for pictorially the calculation in Theorem 51, to give $(\theta \otimes A); (C \otimes \mathrm{ev})$, is



Cutting separately on the X and Z ports we see that this is the same ternary relation P again. □

8. Powerlocales

In this section we show how the upper and lower powerlocales may be constructed on entailment systems. (In the reflexive case, [4] already give an account for the Vietoris powerlocale, which subsumes the upper and lower.)

Recall that if X is a locale then its upper and lower powerlocales $P_U X$ and $P_L X$ are defined by

$$\begin{aligned} \Omega P_U X &= \mathbf{Fr}\langle \Omega X \text{ (qua preframe)}\rangle \\ &= \mathbf{Fr}\langle \Box a \ (a \in \Omega X) \mid \Box \text{ preserves finite meets, directed joins}\rangle, \\ \Omega P_L X &= \mathbf{Fr}\langle \Omega X \text{ (qua suplattice)}\rangle \\ &= \mathbf{Fr}\langle \Diamond a \ (a \in \Omega X) \mid \Diamond \text{ preserves all joins}\rangle. \end{aligned}$$

8.1. The upper powerlocale

We first show that P_U extends to a functor on $\mathbf{SLCLoc}_U$, in other words it is functorial on upper relations. Suppose $f : \Omega Y \to \Omega X$ is a preframe homomorphism.

Then $f;\Box : \Omega Y \to \Omega P_U X$ is a preframe homomorphism and so extends to a frame homomorphism $\bar{f} : \Omega P_U Y \to \Omega P_U X$, defined by $\bar{f}(\Box a) = \Box f(a)$.

$$\begin{array}{ccc} \Omega Y & \xrightarrow{\Box} & \Omega P_U Y \\ f \big\downarrow & & \big\downarrow \bar{f} \\ \Omega X & \xrightarrow[\Box]{} & \Omega P_U X \end{array}$$

This is functorial.

Let us write F for the corresponding endofunctor of $\mathbf{SCFr}_U$, $F(f) = \bar{f}$.

Now suppose we have an entailment system $(X, \vdash)$ corresponding to an idempotent preframe endomorphism j on $\mathbf{Fr}\langle X \rangle$. Since the corresponding stably continuous frame A splits j, we see that $F(A)$ splits the endomorphism $F(j)$ on $F(\mathbf{Fr}\langle X \rangle)$. Now

$$\begin{aligned} \mathbf{Fr}\langle X \rangle &\cong \mathbf{Fr}\langle \mathscr{F} X \text{ (qua } \cup = \vee\text{-semilattice)}\rangle \\ &\cong \mathbf{PreFr}\langle \mathscr{F} X \text{ (qua poset under } \subseteq)\rangle \end{aligned}$$

so

$$F(\mathbf{Fr}\langle X \rangle) \cong \mathbf{Fr}\langle \mathscr{F} X \text{ (qua poset under } \subseteq)\rangle.$$

with V on the right corresponding to $\Box(\bigvee V)$.

This frame is plainly a quotient of $\mathbf{Fr}\langle \mathscr{F} X \rangle$ by a frame homomorphism q^*, and so we get $F(A)$, the frame for the upper powerlocale on $\mathrm{Spec}(X, \vdash)$, by splitting

$$\begin{array}{ccc} \mathbf{Fr}\langle \mathscr{F} X \rangle & \xrightarrow{q^*} & \mathbf{Fr}\langle \mathscr{F} X \text{ (qua poset under } \subseteq)\rangle \\ & & \downarrow F(j) \\ \mathbf{Fr}\langle \mathscr{F} X \rangle & \xleftarrow[q_*]{} & \mathbf{Fr}\langle \mathscr{F} X \text{ (qua poset under } \subseteq)\rangle \end{array}$$

q^* is the frame homomorphism defined by $U \mapsto U$, and q_* is its right adjoint, which we shall show to be a preframe homomorphism.

We have

$$\begin{aligned} \mathbf{Fr}\langle \mathscr{F} X \text{ (qua poset under } \subseteq)\rangle &\cong \mathbf{Fr}\langle \mathscr{F}\mathscr{F} X / \sqsubseteq_L \text{ (qua } \cup = \vee\text{-semilattice)}\rangle \\ &\cong \mathbf{PreFr}\langle \mathscr{F}\mathscr{F} X / \sqsubseteq_L \text{ (qua poset)}\rangle \end{aligned}$$

where $\mathscr{U} \sqsubseteq_L \mathscr{V}$ iff for every $U \in \mathscr{U}$ there is some $V \in \mathscr{V}$ with $U \subseteq V$. We can define a preframe homomorphism

$$\begin{aligned} q_* &: \mathbf{PreFr}\langle \mathscr{F}\mathscr{F} X / \sqsubseteq_L \text{ (qua poset)}\rangle \to \mathbf{Fr}\langle \mathscr{F} X \rangle \\ q_*(\mathscr{U}) &= \textstyle\bigvee^{\uparrow} \{ \bigvee \mathscr{V} \mid \mathscr{V} \sqsubseteq_L \mathscr{U} \}. \end{aligned}$$

We have $q_*; q^* = \mathrm{Id}$, for

$$q^* \circ q_*(\mathscr{U}) = \textstyle\bigvee^{\uparrow} \{ \mathscr{V} \mid \mathscr{V} \sqsubseteq_L \mathscr{U} \} = \mathscr{U}.$$

For $q^*;q_* \geqslant \mathrm{Id}$, we must check it on the preframe basic elements $\bigvee \mathscr{U}$:

$$\bigvee\mathscr{U} \mapsto \mathscr{U} \mapsto \bigvee^{\uparrow} \{\bigvee\mathscr{V} \mid \mathscr{V} \sqsubseteq_L \mathscr{U}\} \geqslant \bigvee\mathscr{U}.$$

Hence the preframe homomorphism q_* is the right adjoint of q^*.

We now turn to calculating the entailment system $(\mathscr{F}X, \vdash_U)$ that corresponds to $q^*; F(j); q_*$. By Proposition 25 we have

$$\mathscr{U} \vdash_U \mathscr{V} \quad \text{iff} \quad \bigwedge\mathscr{U} \leqslant q_* \circ F(j) \circ q^* (\bigvee\mathscr{V}).$$

This right-hand condition is equivalent to $q^*(\bigwedge \mathscr{U}) \leqslant F(j) \circ q^*(\bigvee \mathscr{V})$, which in turn is equivalent to

$$\bigwedge\mathscr{U} \leqslant \bigvee_{V \in \mathscr{V}} F(j) \circ q^*(V). \tag{*}$$

($F(j)$ and q^* are both frame homomorphisms.)

The points of the locale for $\mathbf{Fr}\langle\mathscr{F}X$ (qua poset under $\subseteq$)$\rangle$ are the upper closed subsets of $\mathscr{F}X$. Consider in particular the point $\uparrow\mathscr{U}$, the upper closure of $\mathscr{U}$. It is in $\bigwedge \mathscr{U}$, and hence also in any open $a \geqslant \bigwedge \mathscr{U}$. Conversely, we show that if $\uparrow\mathscr{U}$ is in a then $\bigwedge \mathscr{U} \leqslant a$. It suffices to show this for a subbasic open $V \in \mathscr{F}X$. The point $\uparrow\mathscr{U}$ is in this open iff $V \supseteq U$ for some $U \in \mathscr{U}$, and in this case we see $\bigwedge \mathscr{U} \leqslant U \leqslant V$. It follows that (*) holds iff $\bigwedge \mathscr{U} \leqslant F(j) \circ q^*(V)$ for some $V \in \mathscr{V}$.

In terms of the generators using $\Box$, we have $F(j) \circ q^*(V) = F(j)(\Box \bigvee V) = \Box(j(\bigvee V))$. But $\Box$ is right adjoint to $\Omega\uparrow$, where $\uparrow$ is the unit of the monad P_U, and so our condition reduces to $\Omega\uparrow(\bigwedge \mathscr{U}) \leqslant j(\bigvee V)$, i.e.

$$\bigwedge_{U \in \mathscr{U}} \bigvee U \leqslant j(\bigvee V).$$

Applying the distributivity law on the left, this reduces to finding some $\mathscr{U}' \bowtie \mathscr{U}$ with $\mathscr{U}' \overline{\vdash} \{V\}$.

We have now proved

Theorem 52. *Let* $(X, \vdash)$ *be an entailment system. Then* $P_U(\mathrm{Spec}(X, \vdash))$ *is presented by an entailment system* $(\mathscr{F}X, \vdash_U)$, *where*

$$\mathscr{U} \vdash_U \mathscr{V} \quad \textit{iff} \quad \exists \mathscr{U}', \ \exists V \in \mathscr{V}.\ \mathscr{U} \bowtie \mathscr{U}' \overline{\vdash} \{V\}.$$

8.2. The lower powerlocale

We now turn to the lower powerlocale. The upper powerlocale is intimately bound up with preframe homomorphisms and hence with the whole machinery of entailment systems, so it is remarkable that the lower is calculated by a similar approach to that used for the upper. In fact, it is dual: for a stably locally compact locale X, we have $P_L X \cong (P_U X^\circ)^\circ$. (This is already known classically for stably locally compact spaces, and has been presented for instance by Julian Webster. We have not however found a published reference.) In our constructive localic setting it is obvious once the entailment system machinery is in place.

We first show that P_L extends to a functor on $\mathbf{SLCLoc}_U$, in other words it is functorial on upper relations. Suppose $f:\Omega Y \to \Omega X$ is a preframe homomorphism. By [25] (using a similar technique to that used for the preframe coverage theorem in [9]) we have

$$\begin{aligned}\Omega P_L Y &= \mathbf{Fr}\langle \Omega Y \text{ (qua suplattice)}\rangle\\ &\cong \mathbf{PreFr}\langle \Omega Y \text{ (qua dcpo)}\rangle.\end{aligned}$$

Then $f;\Diamond:\Omega Y \to \Omega P_L X$ is a dcpo morphism and so extends to a preframe homomorphism $\bar{f}:\Omega P_L Y \to \Omega P_L X$, defined by $\bar{f}(\Diamond a) = \Diamond f(a)$.

$$\begin{array}{ccc}\Omega Y & \xrightarrow{\Diamond} & \Omega P_L Y\\ f\big\downarrow & & \big\downarrow \bar{f}\\ \Omega X & \xrightarrow[\Diamond]{} & \Omega P_L X\end{array}$$

This is functorial.

Let us write G for the corresponding endofunctor of $\mathbf{SCFr}_U$, $G(f) = \bar{f}$.

If, with the notation of Subsection 8.1, we have $(X,\vdash)$, j and A splitting j, then $G(A)$ splits $G(j)$. Now

$$\begin{aligned}\mathbf{Fr}\langle X\rangle &\cong \mathbf{Fr}\langle \mathscr{F}X \text{ (qua } \cup = \wedge\text{-semilattice)}\rangle\\ &\cong \mathbf{SupLat}\langle \mathscr{F}X \text{ (qua poset under } \supseteq)\rangle\end{aligned}$$

so

$$G(\mathbf{Fr}\langle X\rangle) \cong \mathbf{Fr}\langle \mathscr{F}X \text{ (qua poset under } \supseteq)\rangle.$$

with V on the right corresponding to $\Diamond(\bigwedge V)$.

Hence we get $G(A)$, the frame for the lower powerlocale, by splitting

$$\begin{array}{ccc}\mathbf{Fr}\langle \mathscr{F}X\rangle & \xrightarrow{q^*} & \mathbf{Fr}\langle \mathscr{F}X \text{ (qua poset under } \supseteq)\rangle\\ & & \downarrow G(j)\\ \mathbf{Fr}\langle \mathscr{F}X\rangle & \xleftarrow[q_*]{} & \mathbf{Fr}\langle \mathscr{F}X \text{ (qua poset under } \supseteq)\rangle\end{array}$$

q^* is the frame homomorphism defined by $U \mapsto U$. Again we can calculate its right adjoint q_* as a preframe homomorphism. We have

$$\begin{aligned}\mathbf{Fr}\langle \mathscr{F}X \text{ (qua poset under } \supseteq)\rangle &\cong \mathbf{Fr}\langle \mathscr{F}\mathscr{F}X/\sqsubseteq_L \text{ (qua } \cup = \vee\text{-semilattice)}\rangle\\ &\cong \mathbf{PreFr}\langle \mathscr{F}\mathscr{F}X/\sqsubseteq_L \text{ (qua poset)}\rangle\end{aligned}$$

where this time $\mathscr{U} \sqsubseteq_L \mathscr{V}$ iff for every $U \in \mathscr{U}$ there is some $V \in \mathscr{V}$ with $U \supseteq V$ (instead of $U \subseteq V$). We can define a preframe homomorphism

$$\begin{aligned}q_* &: \mathbf{PreFr}\langle \mathscr{F}\mathscr{F}X/\sqsubseteq_L \text{ (qua poset)}\rangle \to \mathbf{Fr}\langle \mathscr{F}X\rangle\\ q_*(\mathscr{U}) &= \bigvee\nolimits^{\uparrow} \{\bigvee \mathscr{V} \mid \mathscr{V} \sqsubseteq_L \mathscr{U}\}.\end{aligned}$$

and much as before q_* is right adjoint to q^*.

We now turn to calculating the entailment system $(\mathscr{F}X, \vdash_L)$ that corresponds to $q^*; G(j); q_*$. We find

$$\mathscr{U} \vdash_L \mathscr{V} \quad \text{iff} \quad \bigwedge\mathscr{U} \leqslant G(j)(\bigvee\mathscr{V}).$$

In terms of the generators with $\diamondsuit$,

$$\begin{aligned} G(j)(\bigvee\mathscr{V}) &= G(j)\left(\bigvee_{V\in\mathscr{V}} \diamondsuit(\bigwedge V)\right) \\ &= G(j)\left(\diamondsuit\left(\bigvee_{V\in\mathscr{V}} \bigwedge V\right)\right) \\ &= \diamondsuit\left(j\left(\bigvee_{V\in\mathscr{V}} \bigwedge V\right)\right). \end{aligned}$$

The points of the locale for $\mathbf{Fr}\langle\mathscr{F}X$ (qua poset under $\supseteq$)$\rangle$ are the lower closed subsets of $\mathscr{F}X$. Consider in particular the point $\downarrow\mathscr{U}$, the lower closure of $\mathscr{U}$. It is in the open $\bigwedge\mathscr{U}$, and hence also in any open $b \geqslant \bigwedge\mathscr{U}$. Now every open $a \in \mathbf{Fr}\langle X\rangle$ is a join of elements $\bigwedge W$, and so every $\diamondsuit a$ is (considered as an element of $\mathbf{Fr}\langle\mathscr{F}X$ (qua poset under $\supseteq$)$\rangle$) a join of elements W. In particular, if $\bigwedge\mathscr{U} \leqslant \diamondsuit a = \bigvee_i W_i$ then $\downarrow\mathscr{U}$ is in some W_i, i.e. $W_i \subseteq$ some $U \in \mathscr{U}$ and so $U \leqslant W_i$. Hence, if $\bigwedge\mathscr{U} \leqslant \diamondsuit a$ then $U \leqslant \diamondsuit a$ for some $U \in \mathscr{U}$. Applying $\Omega\downarrow$, where $\downarrow$ is the unit of the lower powerlocale monad, we get $\bigwedge U \leqslant a$. Conversely, if $\bigwedge U \leqslant a$ then by applying $\diamondsuit$ we get $U \leqslant \diamondsuit a$ and so $\bigwedge\mathscr{U} \leqslant \diamondsuit a$.

We can now deduce that

$$\mathscr{U} \vdash_L \mathscr{V} \quad \text{iff} \quad \exists U \in \mathscr{U}.\ \bigwedge U \leqslant j\left(\bigvee_{V\in\mathscr{V}} \bigwedge V\right).$$

Applying the distributivity law on the right, this reduces to finding some $\mathscr{V}' \bowtie \mathscr{V}$ with $\{U\} \overline{\vdash} \mathscr{V}'$.

We have now proved

Theorem 53. *Let* $(X, \vdash)$ *be an entailment system. Then* $P_L(\mathrm{Spec}(X, \vdash))$ *is presented by an entailment system* $(\mathscr{F}X, \vdash_L)$, *where*

$$\mathscr{U} \vdash_L \mathscr{V} \quad \textit{iff} \quad \exists U \in \mathscr{U},\ \exists \mathscr{V}'.\ \{U\} \overline{\vdash} \mathscr{V}' \bowtie \mathscr{V}.$$

Combining this with the corresponding result for the upper powerlocale, and using the fact that duality is got by reversing entailment relations, we have

Theorem 54. *Let X be a stably locally compact locale. Then*

$$P_L X \cong (P_U X^\circ)^\circ.$$

9. Conclusions

We have shown how stably locally compact locales can be captured by predicative geometric structure akin to information systems in a generalized way. The proofs are constructive, using choice free principles based on validity in toposes: in particular, we do not rely on the classical spatiality results for stably locally compact locales.

The work generalizes the reflexive sequent calculus used in [4] to deal with distributive lattices. It develops the multilingual sequent calculus MLS of [13] and [10], using its cut composition and again bringing out the logical flavour, but working in a very bare syntax that does without explicit connectives for conjunction and disjunction.

Comparing our approach with MLS, there seem to be two main differences of mathematical substance. The first is the role of inductive generation. Our tokens are generators for the MLS tokens, which are formed as expressions using meet and join. This in itself is not a big difference, since the MLS papers also discuss the use of generating sets of tokens, with entailment generated by proof rules. However, a difference arises in the approach to constructions such as product, powerdomains and so on. MLS shows how to make generators for the new constructed system out of general tokens (with meets and joins) for the old systems. Thus when constructions are composed there has to be a step of generating general tokens. By contrast in our approach new generators and entailment relations are defined directly out of old generators and entailment relations. This requires more work, but gives completer information and in particular we conjecture it can provide decidability proofs for the inductive construction in MLS. The other main difference is the assumption in MLS of interpolants, allowing them to use Gentzen's cut rule. It remains to be seen whether our axiomatic economy compensates for the greater difficulty in working entirely without Gentzen's cut rule, but we have already seen how use of the cut calculus wraps up quite complicated applications of the distributive law. Duality of stably locally compact locales comes out in a particularly simple way from the symmetry of the syntax.

One question we have not yet been able to answer is that of how to construct the patch locale in terms of entailment systems. (Moshier has announced a simple construction for it in MLS.) If X is a stably locally compact locale, then its patch locale $\mathrm{Patch}(X)$ has been described in [3]. More conveniently, [5] shows that the opens of $\mathrm{Patch}(X)$ are in bijection with the perfect nuclei on ΩX. Following what happens with the reflexive (distributive lattice) case, where one is constructing the free Boolean algebra, one might conjecture that the patch for an entailment system $(X,\vdash)$ can be described as follows—in fact this is the basis of Moshier's construction. Let $Y = X \cup \{\bar{x} \mid x \in X\}$. Then $\mathscr{F}Y \cong \mathscr{F}X \times \mathscr{F}X$ and we shall write the finite subsets of Y in the form $U + \bar{V}$, meaning $U \cup \{\bar{v} \mid v \in V\}$. Thinking of $\bar{v}$ as a Boolean negation $\neg v$, one might attempt to define:

$$U_1 + \overline{V_1} \vdash U_2 + \overline{V_2} \quad \text{iff} \quad U_1 \cup V_2 \vdash U_2 \cup V_1.$$

However, this does not in general define an entailment relation, as can be seen from Example 5. One can calculate that the corresponding entailment system for $X = \{a, b\}$,

$U \vdash V$ iff $\bigwedge U \leqslant j(\bigvee V)$, can be completely listed as

$$\begin{array}{llll} \emptyset \vdash \{a,b\} & \{a\} \vdash \{a,b\} & \{b\} \vdash \{a,b\} & \{a,b\} \vdash \{a,b\} \\ \{a\} \vdash \{a\} & \{a,b\} \vdash \{a\} & & \\ \{a\} \vdash \{b\} & \{a,b\} \vdash \{b\} & & \end{array}$$

Now consider the attempted entailment on $X + \bar{X}$. This is not an entailment relation, for we have

$$\emptyset \vdash \{a, \bar{a}\} \qquad \{a\} \vdash \{b\} \qquad \{\bar{a}\} \vdash \{b\}$$

and hence $\emptyset(\vdash \dagger \vdash)\{b\}$. However, we do not have $\emptyset \vdash \{b\}$.

We hope that a solution to this problem may arise out of a better understanding of the "cut calculus", for example of how an entailment morphism behaves when two of its ports are cut together.

References

[1] S. Abramsky, Domain theory in logical form, Ann. Pure Appl. Logic 51 (1991) 1–77.
[2] S. Abramsky, S.J. Vickers, Quantales, observational logic and process semantics, Math. Struct. Comput. Sci. 3 (1993) 161–227.
[3] B. Banaschewski, G.C.L. Brümmer, Stably continuous frames, Math. Proc. Cambridge Philos. Soc. 104 (1988) 7–19.
[4] J. Cederquist, T. Coquand, Entailment relations and distributive lattices, in: S.R. Buss, P. Hájek, H. Pudlák (Eds.), Logic Colloq. '98, Lecture Notes in Logic, Vol. 13, Association of Symbolic Logic, Poughkeepsie, NY, 2000, pp. 110–123.
[5] M. Escardó, The regular-locally-compact coreflection of a stably locally compact locale, J. Pure Appl. Algebra 157 (1) (2001) 41–55.
[6] P.T. Johnstone, Stone spaces, Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press, Cambridge, 1982.
[7] P.T. Johnstone, Open locales and exponentiation, Contemp. Math. 30 (1984) 84–116.
[8] P.T. Johnstone, Sketches of an Elephant: A Topos Theory Compendium, Vol. 2, Oxford Logic Guides, Vol. 44, Oxford University Press, Oxford, 2002.
[9] P.T. Johnstone, S.J. Vickers, Preframe presentations present, in: A. Carboni, M.C. Pedicchio, G. Rosolini (Eds.), Category Theory—Proc. Como 1990, Lecture Notes in Mathematics, Vol. 1488, Springer, Berlin, 1991, pp. 193–212.
[10] A. Jung, M. Kegelmann, M.A. Moshier, Multilingual sequent calculus and coherent spaces, Fund. Inform. 37 (1999) 369–412.
[11] A. Jung, M. Kegelmann, M.A. Moshier, Stably compact spaces and closed relations, in: S. Brookes, M. Mislove (Eds.), 17th Conf. on Mathematical Foundations of Programming Semantics, Electronic Notes in Theoretical Computer Science, Vol. 45, Elsevier, Amsterdam, 2001, 24 pp.
[12] A. Jung, P. Sünderhauf, On the duality of compact vs. open, in: S. Andima, R.C. Flagg, G. Itzkowitz, P. Misra, Y. Kong, R. Kopperman (Eds.), Papers on General Topology and Applications: Proc. of Eleventh Summer Conf. at University of Southern Maine, Annals of the New York Academy of Sciences, Vol. 806, New York, 1996, pp. 214–230.
[13] M. Kegelmann, Continuous domains in logical form, Ph.D. Thesis, School of Computer Science, University of Birmingham, 1999.
[14] M.A. Moshier, A. Jung, A logic for probabilities in semantics, in: J. Bradfield (Ed.), Computer Science Logic: 16th Internat. Workshop, CSL2002, Lecture Notes in Computer Science, Vol. 2471, Springer, Berlin, 2002, pp. 216–231.

[15] D. Scott, Domains for denotational semantics, in: M. Nielsen, E.M. Schmidt (Eds.), Automata, Languages and Programming, Lecture Notes in Computer Science, Vol. 140, Springer, Berlin, 1982, pp. 577–613.
[16] M. Smyth, Effectively given domains, Theoret. Comput. Sci. 5 (1977) 257–274.
[17] M.B. Smyth, Stable compactification I, J. London Math. Soc. (2) 45 (1992) 321–340.
[18] C.F. Townsend, Preframe techniques in constructive locale theory, Ph.D. Thesis, Department of Computing, Imperial College, London, 1996.
[19] C.F. Townsend, Localic Priestley duality, J. Pure Appl. Algebra 116 (1997) 323–335.
[20] S.J. Vickers, Topology via Logic, Cambridge University Press, Cambridge, 1989.
[21] S.J. Vickers, Information systems for continuous posets, Theoret. Comput. Sci. 114 (1993) 201–229.
[22] S.J. Vickers, Constructive points of powerlocales, Math. Proc. Cambridge Philos. Soc. 122 (1997) 207–222.
[23] S.J. Vickers, Topical categories of domains, Math. Struct. Comput. Sci. 9 (1999) 569–616.
[24] S.J. Vickers, The double powerlocale and exponentiation: a case study in geometric reasoning, Draft available at http://cs.bham.ac.uk/∼sjv, 2001.
[25] S.J. Vickers, C.F. Townsend, A universal characterization of the double powerlocale, Theoret. Comput. Sci., 2004, this volume; doi: 10.1016/j.tcs.2004.01.034.
[26] K. Viglas, Topos aspects of the extended Priestley duality, Ph.D. Thesis, Department of Computing, Imperial College, London, 2004.